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# A POINT INTERPOLATION METHOD FOR TWO-DIMESIONAL SOLIDS

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## ABSTRACT

A Point Interpolation Method (PIM) is presented for stress analysis for two-dimensional solids. In the PIM, the problem domain is represented by properly scattered points. A technique is proposed to construct polynomial interpolants with delta function property based only on a group of arbitrarily distributed points. The PIM equations are then derived using variational principles. In the PIM, the essential boundary conditions can be implemented with ease as in the conventional Finite Element Methods. The present PIM has been coded in FORTRAN. The validity and efficiency of the present PIM formulation are demonstrated through example problems. It is found that the present PIM is very easy to implement, and very flexible for obtained displacements and stresses of desired accuracy in solids. As the elements are not used for meshing the problem domain, the present PIM opens new avenue to develop adaptive analysis codes for stress analysis in solids and structures.

KEYWORD: Meshless Method; Element Free Method; Interpolation Function; Stress Analysis

# 1. INTRODUCTION

The Finite Element Method (FEM) is a robust and well-established method for modeling of complex problems in applied mechanics and related fields. There are, however, some shortcomings in using FEM. Mesh generation required in the FEM can be a very time-consuming and expensive task. In many problems, such as simulating the growth of cracks with arbitrary paths, the mesh generation has become an extremely burdensome task, as it has to be adaptive in the process of simulation. The root of the problems is the use of 'element'. As long as the element is used, the problems mentioned above will not have an easy solution. Therefore, some element free or meshless methods have been proposed and achieved remarkable progress in recent years. Investigations have been carried out on possible numerical methods where meshes are unnecessary. Nayroles et al.[1] have proposed a technique called Diffuse Element (DE) method. Liu et al.[2] have developed a different class of so called Reproducing Kernel Particle (RKP) method. Belytschko et al.[3,4,5] have proposed a meshless method called Element Free Galerkin (EFG) method and applied the EFG to a large variety of problems.

However, because the meshless methods are relatively new, there exist following major technical problems for stress analysis for solids and structure.

- Difficulties in the implementation of essential of boundary conditions.
- Complexity in algorithms for computing the interpolation functions which makes the methods expensive.

Some strategies have been developed for alleviating the above problems [6,7,8,9].

This paper presents a Point Interpolation Method (PIM), in which only a set of points is used to represent the problem domain. The displacement at a point is interpolated by the displacements in the influence domain of this point. A technique is proposed to construct polynomial interpolation functions with delta function property. The interpolation functions can be obtained very efficiently and the essential boundary conditions can be simply imposed, as is done in the conventional FEM.

The PIM equations are derived using the interpolation functions based on the variational principles. Solving these equations for the displacement field, the stresses and strains can then be computed. A PIM program has been developed in FORTRAN, and numerical examples are presented to demonstrate the convergence, validity and efficiency of the PIM.

## 2. POINT INTERPOLANT

Consider a function $u(x)$ defined in domain $\Omega$ discretized by a set of field nodes. The PIM interpolates $u(x)$ from the surrounding nodes of a point $x_Q$ using polynomials

$$u(x, x_Q) = \sum_{i=1}^{n} p_i(x) a_i(x_Q) = \mathbf{p}^{\mathrm{T}}(x)\mathbf{a}(x_Q) \tag{1}$$

where $p_i(x)$ is monomials in the space coordinates $\mathbf{x}^{\mathrm{T}}=[x, y]$, $n$ is the number of nodes in the neighborhood of $x_Q$, $a_i(x_Q)$ is the coefficient for $p_i(x)$, and corresponding to the given point $x_Q$

$$\mathbf{a}^{\mathrm{T}}(x_Q)=[a_1,\ a_2,\ a_3, .....,\ a_n] \tag{2}$$

The $p_i(x)$ in equation (1) is built utilizing the Pascal's triangle [10] shown in Figure 1, so that the basis is complete. A basis in one dimension is provided by

$$\mathbf{p}^{\mathrm{T}}(x)=[1, x, x^2, x^3, x^4,..., x^n] \tag{3a}$$

A basis in two dimensional domain is provided by

$$\mathbf{p}^{\mathrm{T}}(x)=[1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3,....] \tag{3b}$$

The coefficients $a_i$ in equation (1) can be determined by enforcing equation (1) to be satisfied at the $n$ nodes surrounding point $x_Q$. At node $i$ we can have equation

$$u_i=\mathbf{p}^{\mathrm{T}}(x_i)\mathbf{a} \qquad\qquad i=1\sim n \tag{4}$$

where $u_i$ is the nodal value of $u$ at $x=x_i$. Equation (4) can be written in the following matrix form.

$$\mathbf{u}^{\mathrm{e}}=\mathbf{P}_Q\mathbf{a} \tag{5}$$

where

$$\mathbf{u}^{\mathrm{e}}=[\, u_1, u_2, u_3, ....., u_n]^{\mathrm{T}} \tag{6}$$

$$\mathbf{P}_Q{}^{\mathrm{T}}=[\, \mathbf{p}^{\mathrm{T}}(x_1), \mathbf{p}^{\mathrm{T}}(x_2), \mathbf{p}^{\mathrm{T}}(x_3),....., \mathbf{p}^{\mathrm{T}}(x_n)] \tag{7}$$

From equation (5), we have

$$\mathbf{a}=\mathbf{P}_Q{}^{-1}\,\mathbf{u}^{\mathrm{e}} \tag{8}$$

Hence, we have

$$u(x)=\ \boldsymbol{\phi}(x)\,\mathbf{u}^{\mathrm{e}} \tag{9}$$

where the interpolation function $\boldsymbol{\phi}(x)$ is defined by

$$\boldsymbol{\phi}(x)=\mathbf{p}^{\mathrm{T}}(x)\mathbf{P}_Q^{-1}=[\ \phi_1(x),\ \ \phi_2(x),\ \ \phi_3(x),.....,\ \phi_n(x)] \tag{10}$$

The interpolation function $\phi_i(x)$ obtained through above procedure satisfies

$$\phi_i\,(x=x_i)=1 \qquad i=1\sim n \tag{11a}$$

$$\phi_j\,(x=x_i)=0 \qquad j\neq i \tag{11b}$$

$$\sum_{i=1}^{n}\phi_i\,(x)=1 \tag{12}$$

Therefore, the interpolation functions constructed have delta function property, and the essential boundary conditions can be easily imposed in PIM. Equation (12) ensures the interpolation function to be able to represent the rigid motion.

It is possible that $\mathbf{P}_Q^{-1}$ in equation (10) does not exist in some situations. Figure 2(a) shows 6 nodes in the influence domain of point $x_Q$. The 6 nodes are siting in two lines parallel to the $x$–axis. When six nodes are used, the polynomial can be of complete second order with respect to both $x$ and $y$ coordinates. However, these six nodes shown in Figure 2(a) can not possibly represent a second order polynomial in the $y$ direction, as there are only two distinct $y$ coordinate values in all these six nodes. Therefore, matrix $\mathbf{P}_Q$ obtained using these 6 nodes is not reversible.

This problem can be avoided if a appropriate basis is chosen according to the distribution of nodes. If $x^2y$ is used to replace $y^2$ in the basis for the 6 nodes shown in Figure 2(a), the matrix $\mathbf{P}_Q$ will not be singular. This method is useful for some problems that the appropriate basis for the nodal distribution is easy to be found. However, this method is difficult to be used in some practical problems. The method of moving nodes

by a small distance before computing to avoid the singularity of $\mathbf{P}_Q$ can be used. The moving distance $d_{mi}$ of $i$ node is

$$d_{mi} = \alpha \cdot d_i \tag{13}$$

where, $\alpha$ is a small coefficient chosen as $-0.2 \le \alpha \le 0.2$ in this paper. The $d_i$ is the shortest distance between the node $i$ and neighbor nodes. This procedure is illustrated in Figure 2(b). After moving nodes, the new matrix $\mathbf{P}_Q$ is, in general, not singular and $\mathbf{P}_Q^{-1}$ can be computed.

The other problem is how to define the influence domain for a point $x_Q$. A simple method is drawing a circle of radius $r$. The circle can be defined as the influence domain. The number of nodes, $n$, can be determined by counting all the points in the influence domain. In this paper, we use $n$=4~15.

## 3. VARIATIONAL FORM OF PIM

We consider the following two-dimensional problem of solid mechanics in domain $\Omega$ bounded by $\Gamma$:

$$\nabla \boldsymbol{\sigma} + \mathbf{b} = 0 \qquad \text{in } \Omega \tag{14}$$

where $\sigma$ is the stress tensor, which corresponds to the displacement field $\mathbf{u}=\{u, v\}^{\mathrm{T}}$, and $\mathbf{b}$ is the body force vector. The boundary condition are given as follows:

$$\sigma \cdot \mathbf{n} = \bar{\mathbf{t}} \qquad \text{on the natural boundary } \Gamma_{\mathrm{t}} \tag{15}$$

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on the essential boundary } \Gamma_{\mathrm{u}} \tag{16}$$

in which the superposed bar denotes prescribed boundary values and **n** is the unit outward normal to domain $\Omega$.

The variational (weak) form of the equilibrium equation is posed as follows

$$\int_{\Omega} \delta(\nabla_s \mathbf{u}^T) \cdot \sigma d\Omega - \int_{\Omega} \delta\mathbf{u}^T \cdot \mathbf{b} d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T \cdot \bar{\mathbf{t}} d\Gamma = 0 \tag{17}$$

Substituting the expression of $u$ and $v$ given in equation (9) into the weak form (17) yields

$$\mathbf{Ku}=\mathbf{f} \tag{18}$$

where

$$\mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega \tag{19a}$$

$$\mathbf{f}_i = \int_{\Gamma_t} \phi_i \bar{\mathbf{t}} d\Gamma + \int_{\Omega} \phi_i \mathbf{b} d\Omega \tag{19b}$$

$$\mathbf{B}_i = \begin{bmatrix} \phi_{i,x} & 0 \\ 0 & \phi_{i,y} \\ \phi_{i,y} & \phi_{i,x} \end{bmatrix} \tag{19c}$$

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \text{ for plane stress} \tag{19d}$$

It may be noted that the displacement function $\mathbf{u}(x)$ is always differentiable at any given point. The order of continuity depends on field nodes included in the influence domain of the point. Using 3 field nodes ensures $C^0$ continuity, and using 6 field nodes ensures $C^1$ continuity.

In order to obtain the integrals in equations (19a) and (19b), a cell structure that is independent on the field nodes is used. In each cell, Gauss quadrature is employed. The number of quadrature points depends on the number of nodes in a cell[11]. A detail investigation on the density of the nodes and the density of the Gauss points is reported by Liu and Yan [12]. Stiffness matrix is computed at each quadrature point and are assembled for all the field nodes.

The flowchart of the PIM can be given briefly as follows:

1. loop over cells of domain
   2. loop over quadrature points $x_Q$ in cell $i$
      a. according a defined criterion to determine the domain of influence of $x_Q$ and find the nodes included in the domain ;
      b. compute $\phi_i(x_Q)$ and $\phi_{i,j}(x_Q)$at the quadrature point;
      c. evaluate contributions to equation;
      d. assemble contribution to nodes;
   3. end quadrature point loop
4. end cell loop

# 4. COMPARISON BETWEEN PIM, EFG AND FEM

## 4.1 The Interpolant Procedure

### PIM versus FEM

The interpolant procedure in the PIM is based on a group of arbitrary distributed nodes as discussed above. The interpolant procedure in the FEM is the same as PIM, but based on

an element. In both FEM and PIM, the number of monomials used in the base interpolating functions, *m*, is the same as the number of nodes, *n*, utilized. Therefore, the interpolant functions have the property of delta function.

The interpolation at a sampling point in the PIM is performed over the influence domain of the point, which may overlap with the influence domains of other sampling points. FEM defines the shape functions over pre-defined regions called elements, and there is no overlapping.

**PIM versus EFG**

The interpolant procedure of EFG is also based only on a group of arbitrarily distributed nodes. However, the basis number of $\mathbf{p}(x)$ $m$ is usually different from the number of nodes $n$ in the influence domain, $m \neq n$. The interpolant $u^h(x)$

$$u^h(x) = \sum_{i=1}^{m} p_i(x) a_i = \mathbf{p^T(x)a} \tag{20}$$

$u^h(x_i)$ does not equal $u(x_i)$. The least squares approximation is employed to determine $a(x)$, through minimizing

$$J = \sum_{i}^{n} w(x - x_i)[\mathbf{p}^T(x_i)\mathbf{a}(x) - u(x_i)]^2 \tag{21}$$

where $w(x\text{-}x_i)$ is the weight function. Therefore, the shape function $\phi(x)$ lacks the delta function property. The shape function of EFG is more complicated than PIM. The weight function is different for different problems. It takes extra effort to choose and compute the weight function.

**4.2 Imposition of Essential Boundary Condition**

Because the shape function in PIM and FEM has the delta function property, the essential boundary conditions can be implemented with ease. However, the shape function in EFG lacks the delta function property, the accurate and efficient imposition of essential boundary condition often presents difficulties. Strategies have been developed for alleviating the above problem, such as using the Lagrange multipliers [3], Penalty method[7,9] and methods coupling with FEM [13]. The Lagrange multiplier is considered standard.

In using Lagrange multipliers, the coefficient matrix obtained in EFG is a fully populated matrix. It is also larger in size than the stiffness matrix obtained in PIM, especially when the number of nodes in boundary is large. These make EFG inconvenient and time consuming. In the PIM, the stiffness matrix has the same characters as that obtained in conventional FEM. Many computation techniques developed in FEM can be utilized directly in the PIM.

## 5. NUMERICAL RESULT

### 5.1 Patch Test

The first numerical example is the standard patch tests, shown in Figure 3. Two patches are tested. Figure 3(a) shows a patch with 9 nodes of which one is interior node. Figure 3 (b) shows a patch of 15 nodes including 7 irregular interior nodes. A 2×2 rectangular background cell structure is used for integration in these patch tests.

In these patch tests, the displacements are prescribed on all outside boundaries by a linear function of $x$ and $y$ on the patches of dimension $L_x$=2.0 by $L_y$=2.0. The parameters are taken as $E$=1.0,and $\nu$=0.3. The linear displacement functions are $u_x$=0.6$x$ and $u_y$=0.6$y$.

Satisfaction of the patch test requires that the displacement of any interior node be given by the same linear function and that the strains and stresses be constant in the patch.

In the 9 nodes patch test, a circle of radius $r$=1.2 is defined as the influence domain for a point $x_Q$ . Therefore, 4~6 nodes are usually used in the interpolation. The PIM exactly passes the 9 nodes patch test. In the 15 nodes patch test, since the interior nodes are randomly distributed, 5~6 nodes are used in the interpolation for a point $x_Q$ according to the distances between nodes and the point $x_Q$ . The displacements and stresses of 15 nodes patch test are obtained and listed in Tables 1. It is shown that the PIM passes the patch test exactly. In Figure 3 (b), the nodes 9 and 10 are deliberately placed close to each other. It is found that this does not effect the computational results.

It should be noted that the continuity and consistence of PIM depend on the number of the nodes used in the interpolation and the number of the nodes included in the overlapping area of the neighboring influence domains of two sampling points. In order to guarantee $C^0$ continuity, at least 2 nodes should be included in an overlapping area. An exact numerical integration for the stiffness matrix is also required in a patch test with irregular distributed nodes. Any integration error can lead to a failure in the patch test. For example, in obtaining Tables 1, the rectangular background cells are used to ensure an accurate numerical integration. If the integration is carried out using arbitrary quadrilateral background cells, the numerical integration can not be accurate, and the patch test will fail. This is because the coordinate mapping will result in a Jocobian matrix in the denominator of the integrand in the stiffness matrix integration. The Gauss quadrature can fail to give the exact result for such integration regardless how many Gauss points are used, as the integrand is not a polynomial of any order.

### 5.2 Cantilever beam

The PIM is also applied to analyze stresses in a cantilever beam. The beam is of length $L$ and height $D$ subjected to a parabolic traction at the free end as shown in Figure 4. The beam has a unit thickness and a plane stress problem is considered. The analytical solution is available and can be found in a textbook by Timoshenko and Goodier[14]:

$$u_x = -\frac{Py}{6EI}\left[(6L-3x)x+(2+\nu)\left(y^2-\frac{D^2}{4}\right)\right] \tag{22}$$

$$u_y = -\frac{P}{6EI}\left[3\nu y^2(L-x)+(4+5\nu)\frac{D^2x}{4}+(3L-x)x^2\right] \tag{23}$$

where the moment of inertia $I$ of the beam is given by

$$I=\frac{D^3}{12} \tag{24}$$

The stresses corresponding to the displacements (22) and (23) are

$$\sigma_x(x,y) = -\frac{P(L-x)y}{I} \tag{25}$$

$$\sigma_y = 0 \tag{26}$$

$$\sigma_{xy}(x,y) = \frac{P}{2I}\left[\frac{D^2}{4}-y^2\right] \tag{27}$$

The parameters are taken as $E$=3.0×10$^7$, $\nu$=0.3, $D$=12, $L$=48, and $P$=1000. Both a regular nodal distribution and an irregular nodal distribution shown in Figure 5 are employed. 20×8 background integration cell structure is used. In each integration cell, 4×4 Gauss

quadrature is used to evaluate the stiffness matrix of the PIM. The computation is done using two different sizes of circular influence domains.

Figure 6 shows a comparison of the analytical solution and the present numerical solution for the beam deflection along the *x*-axis. The plot shows an excellent agreement between the analytical and numerical results. Figure 7 illustrates the comparison between the shear stress calculated analytically and using the PIM at the section of $x=L/2$. Again, very good agreement is observed for both regular and irregular nodal distribution.

For the error analysis, we define the energy norm as an error indicator, as the accuracy in strain or stress is much more critical than the displacements.

$$e_e = \{\int_\Omega (\varepsilon^{PIM} - \varepsilon^{EXACT})^T D(\varepsilon^{PIM} - \varepsilon^{EXACT}) d\Omega\}^{1/2} \tag{28}$$

The convergence of PIM is studied. The regular nodal distribution is used for comparison. The investigation is done for $n$=4, 6, 8 and 10. The convergence with mesh refinement is shown in Figure 8. The $h$ is equivalent to the maximum element size in the FEM analysis in this case. It is observed that the convergence of PIM is very good. It maybe also mentioned here that when the field nodes are very dense, the number of nodes used for interpolation, $n$, does not effect the accuracy significantly, if the density of the field nodes is the same. This finding provides flexibility in choosing the number of field nodes for interpolation.

In order to study the efficiency, the PIM is compared with EFG method in the same conditions. It is found that the computational error of EFG is bigger than PIM when the number of node $n$ in the influence domain is same. To achieve the same accuracy, the $n$ used in EFG must be bigger than PIM. Therefore, the comparison is made for two

situations: the same $n$ and the same accuracy. The CPU time of PIM and EFG is shown in Table 2. It is found that PIM uses much less CPU time than EFG.

### 5.3 Hole in an infinite plate

Consider now a plate with a central circular hole subjected to a unidirectional tensile load of 1.0 in the $x$ direction as shown in Figure 9. Due to symmetry, only the upper right quadrant of the plate is modeled. Plane strain condition is assumed, and $E$=1.0×10$^3$, $\nu$=0.3. Symmetry conditions are imposed on the left and bottom edges, and the inner boundary of the hole is traction free. The exact solution for the stresses is

$$\sigma_x(x,y)=1-\frac{a^2}{r^2}\{\frac{3}{2}\cos 2\theta+\cos 4\theta\}+\frac{3a^4}{2r^4}\cos 4\theta \tag{29a}$$

$$\sigma_y(x,y)=-\frac{a^2}{r^2}\{\frac{1}{2}\cos 2\theta-\cos 4\theta\}-\frac{3a^4}{2r^4}\cos 4\theta \tag{29b}$$

$$\sigma_{xy}(x,y)=-\frac{a^2}{r^2}\{\frac{1}{2}\sin 2\theta+\sin 4\theta\}+\frac{3a^4}{2r^4}\sin 4\theta \tag{29c}$$

where $(r,\theta)$ are the polar coordinates and $\theta$ is measured counter-clockwise from the positive $x$ axis. Traction boundary conditions given by the exact solution (29) are imposed on the right ($x$=5) and top ($y$=5) edges.

The circle influence domain is used. The radius $r$ is defined as

$$r=2.0\times d_{\max} \tag{30}$$

where $d_{\max}$ is equivalent to the maximum element size in the FEM analysis in this case. If the number of nodes in the influence domain is more than 15, only 15 nodes with shorter distances to the integration point are used in the interpolation. It is found that for

displacement, results obtained are identical. As the stress is most critical, detail results are presented here. The stress $\sigma_x$ at $x$=0 obtained using the PIM is shown in Figure 10. The result obtained using FEM software PATRAN using the same nodes (regular mesh) as PIM is shown in the same figure. It can be observed from this figure that the PIM yields satisfied results for the problem.

### 5.4 Bridge pier

In this example the PIM is used in stress analysis of a bridge pier subjected five concentrated forces on the top of the pier, as shown in Figure 11. The problem is solved for the plane strain case with $E$=40GPa, $\nu$=0.15 and $p$=100kN. Due to symmetry, only the right half of the pier is modeled. The nodal arrangement is shown in Figure 12(a). The problem is also analyzed by FEM software PATRAN using the mesh shown in Figure 12(b), which has the same number of nodes as PIM.

The displacements at some nodes are listed in Table 3. The results obtained by the present method are in very good agreement with those obtained using FEM. The distribution of stress $\sigma_y$ in the domain obtained by FEM and PIM are shown in Figure 13. It can be found that PIM obtains a satisfied result for this problem.

## 6. DISCUSSION AND CONCLUSIONS

A Point Interpolation Method (PIM) has been presented. In this method a technique is proposed to construct polynomial interpolation function with delta function property using a group of arbitrarily distributed points. The PIM overcomes the drawbacks in other element-free methods, such as the complexity in the shape function, difficult in the

implementation of the essential boundary conditions. Many computation techniques developed in FEM can be utilized in the PIM. Numerical examples have demonstrated effectiveness of the PIM. The method offers a potential numerical alternative simple and efficient procedure to handling problems of industrial applications because it is unnecessary to use elements.

The advantages of the method do not come without some cost. When the PIM shape function is computed, the $\mathbf{P}_Q^{-1}$ may not exist in some situation. Some techniques must be used to overcome this problem, such as moving nodes randomly by small distances and choosing appropriate basis. More research work need be done to avoid the problem. In order to shorten computation time and improve the computation accuracy, especially when the nodes distributed very irregularly, an efficient nodes choosing method and software need be developed.

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Table 1 PIM results at interior nodes located irregularly in the 15 nodes patch

| | **(*x*,*y*)** for interior node | $u_x$ | $u_y$ | $\sigma_x$ | $\sigma_y$ | $\tau_{xy}$ |
|---|---|---|---|---|---|---|
| 9 | **(1.0, 0.0)** | 0.60000 | 0.0000 | .8571428571 | .8571428571 | -3.94E-16 |
| 10 | **(1.1, 0.1)** | 0.66000 | 0.06000 | .8571428571 | .8571428571 | -8.85E-17 |
| 11 | (1.9, 0.3) | 1.14000 | 0.18000 | .8571428571 | .8571428571 | 2.78E-16 |
| 12 | (0.1, 0.5) | 0.06000 | 0.30000 | .8571428571 | .8571428571 | 8.08E-16 |
| 13 | (0.5, 0.2) | 0.30000 | 0.12000 | .8571428571 | .8571428571 | 1.65E-16 |
| 14 | (1.2,-0.9) | 0.72000 | -0.54000 | .8571428571 | .8571428571 | -2.88E-15 |
| 15 | (0.3,-0.8) | 0.18000 | -0.48000 | .8571428571 | .8571428571 | -1.90E-15 |

Table 2 The CPU time of PIM and EFG

| Num. Of nodes | CPU time of EFG(*s*) | | CPU time of PIM (*s*) |
|---|---|---|---|
| | With same *n* | With same accuracy | |
| 55 | 3.3 | 8.4 | 2.1 |
| 189 | 67.2 | 95.1 | 8.2 |
| 561 | 1731.7 | 1818.4 | 32.1 |

Table 3 Vertical displacement of the bridge pier

| Nodes | *X* | *y* | Displacement ($\times 10^5$) | |
|---|---|---|---|---|
| | | | PIM | FEM |
| 1 | 0.0 | 10.0 | -1.6814 | -1.7003 |
| 5 | 0.0 | 30.0 | -2.6252 | -2.6651 |
| 6 | 5.0 | 30.0 | -2.2953 | -2.2828 |
| 7 | 10. | 30.0 | -2.2602 | -2.2725 |