STABILITY OF APPROXIMATIONS OF AVERAGE RUN LENGTH OF RISK-ADJUSTED CUSUM SCHEMES USING THE MARKOV APPROACH: COMPARING TWO METHODS OF CALCULATING TRANSITION PROBABILITIES

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Key Words ARL, risk-adjusted CUSUM, Markov property, medical monitoring, adverse outcomes

ABSTRACT

Risk-adjusted CUSUM schemes are designed to monitor the number of adverse outcomes following a medical procedure. An approximation of the average run length (ARL), which is the usual performance measure for a risk-adjusted CUSUM, may be found using its Markov property. We compare two methods of computing transition probability matrices where the risk model classifies patient populations into discrete, finite levels of risk. For the first method, a process of scaling and rounding off concentrates probability in the centre of the Markov states, which are non-overlapping sub-intervals of the CUSUM decision interval, and, for the second, a smoothing process spreads probability uniformly across the Markov states. Examples of risk-adjusted CUSUM schemes are used to show, if rounding is used to calculate transition probabilities, the values of ARLs estimated using the Markov property vary erratically as the number of Markov states vary and, on occasion, fail to converge for mesh sizes up to 3,000. On the other hand, if smoothing is used, the approximate ARL values remain stable as the number of Markov states vary. The smoothing technique gave good estimates of the ARL where there were less than 1,000 Markov states.
A criterion for cumulative summation (CUSUM) schemes to monitor the number of non-conforming items in industrial manufacturing processes is the probability, \( \pi \), that an item is non-conforming is constant (Hawkins and Olwell, 1998, Page 122). Such schemes are inappropriate for monitoring the number of adverse outcomes following medical procedures because each patient is unique and, clearly, the assumption, that each outcome following a medical procedure has constant probability, is false. The expected probability \( \pi_t \) of an adverse outcome for patient \( t \) may be estimated prior to the procedure using an appropriate risk model. For example, the Parsonnet score (Parsonnet et al., 1989) and the EuroSCORE (Nashef et al., 1999) are used to estimate the probability that a patient who undergoes a cardiac surgical operation will die after the operation, where death is defined, for example, as in-hospital or within 30 days of the operation. In the context of monitoring medical outcomes, a risk model adjusts for variation in the patient population so that any alarm will be due to a change in the quality of treatment (Iezzoni, 1997). The risk-adjusted CUSUM (Steiner et al., 2000) is a scheme which is suited to monitoring adverse medical outcomes because it allows for patient variability.

The risk-adjusted CUSUM scheme signals an alarm when it crosses some predetermined decision boundary, \( h \). It is possible for the CUSUM to signal when there is no shift in the outcome rate. This is a false alarm analogous to a false positive error in hypothesis testing. If there is a shift in the outcome rate, the time taken for a signal after the change occurs is analogous to the power of a hypothesis test. As in hypothesis testing, the performance of the CUSUM is a compromise between the time to false alarms and the time to true alarms (Hawkins and Olwell, 1998) so that the number of false alarms is tolerable but the response to an actual shift in the outcome rate is timely. Two useful but imperfect measures, imperfect because the run length distribution is highly variable (Hawkins and Olwell, 1998), of the performance of a CUSUM are the average run length (ARL) to an alarm when there has been no shift in the outcome rate and the ARL to an alarm after a shift in the outcome rate.

Grigg et al. (2003) note there are at least three ways to determine ARLs for CUSUM
charts. Simulation is the most straightforward. It is time consuming and cumbersome but is useful when particular complexities of a chart, such as risk-adjustment and the discreteness of monitoring, make other approaches difficult. Another approach is to use numerical methods to solve an integral equation (Page, 1954) but, for more complex CUSUMs, complicated integral equations are difficult and, in some instances, impossible to solve. The final approach Grigg et al. (2003) describe is the Markov chain methodology used by Steiner et al. (2000) which provides a particularly convenient way to provide information on a variety of features of the run length distribution, such as the ARL, run length standard deviation, probability of crossing at or before a given time point, $t$, and higher moments. The Markov method requires all real values in the decision interval, $(0, h)$, to be discretized so, ideally, the number of Markov states of the transition probability matrix should be as great as possible to minimize the error of the approximation. However, the degree of discretization is constrained by the computational intensity: the greater the mesh size of the transition matrix the greater the computer time required to manipulate it.

The method of computing the transition probability matrix described by Steiner et al. (2000) is close to that proposed by Brook and Evans (1972). In that method the continuum is approximated by placing the CUSUM $C_{t-1}$ at the centre of an interval $S_{t-1}$, concentrating all the probability at the centre of $S_{t-1}$. Fu et al. (2003) provide an example using this method, which we shall call rounding, and obtain accurate estimates of the ARL after 500 discretizations. On the other hand, Hawkins (1992) warns there is considerable experience that the rounding method of computing the transition probabilities leads to poor accuracy except at very fine discretizations. He proposes a more complex alternative of calculating the transition probabilities by smoothing the probability over the interval for the Markov state. This smoothing method is computationally attractive because it achieves accurate results using fewer discretizations than the rounding approach. Hawkins (1992) also warns that discrete jumps in the cumulative distribution function necessitate a finer mesh for an accurate final answer.

Our purpose is to show that calculation of the transition probability matrix using smooth-
ing provides more accurate approximations of the ARLs of risk-adjusted CUSUMs using the Markov approach than those provided by calculation of transition probabilities using rounding. We only consider risk models where the predicted probabilities of adverse outcomes take a finite number of values. In Section 2 we review the risk-adjusted CUSUM scheme, introduce Hawkins’ method for computing the transition probabilities and adapt it for the risk adjusted CUSUM. In Section 3, we use some examples to show that, if transition probabilities are calculated using smoothing, ARL approximations using the Markov approach remain stable as the number of Markov states varies and converge to a limit as the number of discretizations increases but, if rounding is used, they are unstable and converge more slowly. In Section 4 we discuss the results.

2. METHODS

The risk-adjusted CUSUM is used to monitor for a step increase, from \( p_0 \) to \( p_1 \), in the rate of adverse outcomes in a patient population. It takes the usual form

\[
C_t = \max(0, C_{t-1} + W_t),
\]

where \( C_t \) is the CUSUM at time \( t \), for \( t = 1, 2, \ldots \), and the CUSUM weight, \( W_t = \log \left\{ l(y, p_1)/l(y, p_0) \right\} \), is the scoring found using the sequential-likelihood ratio test (Page, 1954). Observations, \( Y \), are assumed independent and there is an alarm if \( C_t \geq h \), where \([0, h)\) is the decision interval. It is possible to commence monitoring at any \( 0 \leq C_0 < h \) but we restrict our discussion to monitoring schemes that commence at \( C_0 = 0 \). In the context of medical procedure outcomes used for the discussion in this paper, the risk-adjusted CUSUM \( C_t \) relates to patient \( t \).

During the assessment prior to undergoing a medical procedure, the patient’s risk of an adverse outcome, \( X_t \) for patient \( t \), is scored by the medical practitioner who will undertake the procedure. A typical score is a finite, ordered scale of risk which takes integer values, \( x \) for \( x = 0, 1, \ldots, x_{\text{max}} \), where \( x_{\text{max}} \) is the largest value that the risk score takes. As shown in Section 2.1, the risk score may be used to estimate the expected probability \( p_t \) that patient \( t \) will experience an adverse outcome. When used to monitor the number of adverse outcomes,
the risk-adjusted CUSUM allows for the varying \( p_t \) of the patient population by sequentially testing the hypotheses,

\[
H_0 : \frac{\pi_0/(1-\pi_0)}{p_t/(1-p_t)} = R_0,
\]

where \( \pi_0 \) is the probability of an adverse outcome for an in-control process and \( R_0 \) is the odds ratio of the odds of an adverse outcome for an in-control process to the expected odds of an adverse outcome after risk assessment of patient \( t \), versus

\[
H_1 : \frac{\pi_1/(1-\pi_1)}{p_t/(1-p_t)} = R_A,
\]

where \( \pi_1 \) is the probability of an adverse outcome for an out-of-control process and \( R_A \) is the odds ratio of the odds of an adverse outcome for an out-of-control process to the expected odds of an adverse outcome after risk assessment of patient \( t \).

From Steiner et al. (2000, Equation (2.3)), the weight \( W_t \) for observation \( Y_t \) of a risk-adjusted CUSUM scheme is

\[
W_t = \log \left[ \left\{ \frac{1-(1-R_0)p_t}{1-(1-R_A)p_t} \right\}^{y_t} \left\{ \frac{1-(1-R_0)p_t}{1-(1-R_A)p_t} \right\}^{1-y_t} \right], \quad y_t \in \{0, 1\}. \tag{2}
\]

2.1. CONDITIONAL DISTRIBUTION OF \( C_t \)

As Hawkins and Olwell (1997, Page 152) note, it follows from the assumed independence of the observations \( Y \) and the recursive definition of the CUSUM \( C_t \), given in Equation (1), that

\[
\Pr(C_t \mid C_0, C_1, \ldots, C_{t-1}) = \Pr(C_t \mid C_{t-1}).
\]

Thus the conditional probability of \( C_t \) is given by

\[
\Pr(C_t \mid C_{t-1}) = \Pr(W_t).
\]

From Equation (2), the event \( W_t = w \) is a function of the parameters \( R_0 \) and \( R_A \) and the random events that the expected probability of an adverse outcome \( p_t = p \) and the outcome \( Y_t = y_t \). Thus \( \Pr(W_t = w) \) is given by \( \Pr(Y_t = y_t, p_t = p) = \Pr(Y_t = y_t \mid p_t) \Pr(p_t = p) \). We assume that observed outcomes are distributed \( Y_t \sim \text{Bernoulli}(p_t) \) so that

\[
\Pr(Y_t \mid p_t) = p_t^{y_t} (1-p_t)^{1-y_t}.
\]
Risks that patients will experience an adverse outcome are expressed on a linear scale, so a risk score $X_t = x$ that patient $t$ will experience an adverse outcome does not accurately reflect the probability of an adverse event. A reasonable model for the expected probability of mortality $p_t$ of the $t^{th}$ patient is given by

$$\text{logit}(p_t) = \alpha + \beta x,$$

where $(\alpha, \beta)^T$ is the regression parameter. For example, Steiner et al. (2000) used this model to calibrate the Parsonnet score for the expected probability of mortality for their example of the risk-adjusted CUSUM to monitor patient mortality following cardiac surgery. Now the patient population has some discrete distribution $\Pr(X_t = x)$. Hence

$$\Pr\{\text{logit}(p_t) = \alpha + \beta x\} = \Pr(X_t = x).$$

Therefore the probability distribution of $W_t$ is given by

$$\Pr(W_t = w) = p_t y_t (1 - p_t)^{1 - y_t} \Pr(X_t = x),$$

where the value of $w$ for patient $t$, with risk score $x$ and outcome $y_t$, is computed by substituting the value of $p_t$, calculated using Equation (3), and the value of $y_t$ into Equation (2).

The Markov transition probabilities are found from the joint distribution of the observed mortalities, $y_t$, and the risk scores, $X_t$. We note that, for such a risk adjusted CUSUM, the weights are discrete, finite and take irrational values, $(w_1, \ldots, w_N)$, with a probability distribution $(v_1, \ldots, v_N)$ where $\Pr(W_t = w_n) = v_n$ for $n = 1, \ldots, N$. If the values of $W$ are ordered so that

$$w_1 < w_2 < \cdots < w_n < \cdots < w_N$$

the cumulative probability distribution $\Pr(W \leq w_n) = V_n$ is given by $\sum_{j=1}^{n} v_j$.

2.2. CALCULATING TRANSITION PROBABILITIES USING SMOOTHING

When using the Markov property of CUSUMs to approximate their ARLs, the state space of the CUSUM, $C_t$, is discretized into $M + 1$ states commencing at a reflecting state 0. State
M is an absorbing state equivalent to $C_t \geq h$. The width, $\Delta$, of States $1$ to $M-1$, which are non-overlapping intervals in $(0, h)$, is $\Delta = h/M$.

Brook and Evans (1972) show that the ARL may be found by solving the equation

$$E(\lambda) = (I - R)^{-1}1,$$

where $E(\lambda)$ is the $M \times 1$ vector of expected run lengths to a signal, $I$ is the $M \times M$ identity matrix, $R$ is the sub-matrix of the transition probability matrix excluding transitions from or to the absorbing state, and $1$ is an $M \times 1$ vector with each element $1$.

Steiner et al (2000, Appendix) give a method of scaling and rounding off to calculate the transition probabilities of the matrix $R$ for risk-adjusted CUSUMs monitoring populations with discrete and finite categories of risk.

Smoothed transition probabilities may be computed using the equation

$$\Pr(a < S_n < b \mid c < S_{n-1} < d) = \int_c^d \{F(b - s) - F(a - s)\} d\mu(s)$$

given in Hawkins and Olwell (1997, Page 155). For a CUSUM moving from interval $S_{t-1} = i$ to interval $S_t = j$ we let $\mu(x)$ be the distribution function of $S_{t-1}$ conditional on $(i-1)\Delta < S_{t-1} < i\Delta$ and we have $V(x)$ as the cumulative distribution of $W$. Then Equation (5) becomes

$$\Pr[(j-1)\Delta < S_t < j\Delta \mid (i-1)\Delta < S_{t-1} < i\Delta] = \int_{(i-1)\Delta}^{i\Delta} \{V[j\Delta - s] - V[(j-1)\Delta - s]\} d\mu(s).$$

Assume $\mu$ to be uniform so that $d\mu(s) = ds/\Delta$. For a transition from the $i^{th}$ to the $j^{th}$ Markov state we must have $(i-1)\Delta + w_n > (j-1)\Delta$ or $i\Delta + w_n < j\Delta$. Suppose $(i-1)\Delta + w_n > (j-1)\Delta$ then we define $f \in (0, 1)$ such that

$$f = \frac{w_n - (j-i)\Delta}{\Delta}.$$ 

The cumulative distribution $V(x)$ has a discontinuity at $w_n = (j-i+f)\Delta$ where it steps
by \( v_n \) from \( V_{n-1} \) to \( V_n \). Hence, Equation (5) may be evaluated as

\[
\Pr(S_t = j \mid S_{t-1} = i, W_n) = \int_{(i-1)\Delta}^{(i-1+f)\Delta} \{V[j\Delta - s] - V[(j-1)\Delta - s]\} \, ds / \Delta \\
+ \int_{(i-1+f)\Delta}^{i\Delta} \{V[j\Delta - s] - V[(j-1)\Delta - s]\} \, ds / \Delta \\
= (1 - f)v_n
\]

It is possible that, for a step of size \( w_n \), the transition is from \( S_{t-1} = i \) to \( S_t = j + 1 \). Then we find that Equation (6) evaluates as

\[
\Pr(S_t = j + 1 \mid S_{t-1} = i, W_n) = fv_n
\]

There are special cases. If the transition from the \( i \)th to the \( j \)th Markov state is such that \((i-1)\Delta + w_n > h\), then all \( v_n \), the probability associated with the event \( W_t = w_n \), accumulates with the transition probability \( \Pr(S_t = M \mid S_{t-1} = i) \), where \( S_M \) is the absorbing state, and if the transition is such that \( i\Delta + w_n < 0 \), then all \( v_n \) accumulates with \( \Pr(S_t = 0 \mid S_{t-1} = i) \), where \( S_0 \) is the reflecting boundary.

2.3. SIMULATION METHOD

The process of simulating run lengths of risk-adjusted CUSUMs schemes is by

- drawing a risk score at random from the population of scores;

- computing each patient’s probability, \( p_t \), of an adverse outcome; for example, if the risk model is that the expected probability \( p_t \) of an adverse outcome and the risk score \( x_t \) have a logit relationship, \( p_t \) is computed according to Equation (3);

- randomly generating outcome events \( y_t \in \{0, 1\} \)
  
  - for an in-control process, let \( R_0 = 1 \), then \( \Pr(Y_t = 0) = 1 - p_t \) and \( \Pr(Y_t = 1) = p_t \), or
  
  - for an out-of-control process, \( \Pr(Y_t = 0) = (1 - p_t) / \{1 + (R_A - 1)p_t\} \) and \( \Pr(Y_t = 1) = R_A p_t / \{1 + (R_A - 1)p_t\} \).
monitoring the outcomes with the risk-adjusted CUSUM, with decision threshold $h$, tuned to signal if the odds ratio of observed to expected outcomes is $R_A$;

- recording the run length to a signal; and

- repeating until the prescribed number of run lengths recorded.

3. RESULTS

An artificial study, Study 1, illustrates the method under an unrealistic risk score distribution. Suppose a risk-adjusted CUSUM scheme is used to monitor the number of adverse outcomes following a medical procedure where the patient population may be stratified so that the risk of an adverse outcome has twenty-four discrete probabilities, 0.04, 0.08, \ldots, 0.96, with uniform distribution. Approximate ARLs using the Markov approach were found where the decision threshold $h$ was set at 4.5 and the process out-of-control, and where $h$ was 3.5, 4.5 and 5.5 and the process in-control. Discretization of the decision interval $(0, h)$ increased by increments of 10 to a maximum of 3,000 Markov states.

The respective plots of the approximate ARL against the number of discretizations are shown in Rows A, B, C, and D of Figure 1. Column 1 gives ARLs computed for the transition probabilities calculated using rounding and Column 2 gives ARLs for transition probabilities found using smoothing. The dashed or dotted lines on the respective plots in Columns 1 or 2, which are the lower and upper 95% confidence limits for the ARL of 100,000 run lengths simulated using the method described in Section 2.3, are used as a benchmark for the accuracy of the approximations found using the Markov approach. Simulations were done using the R statistical application (R Project, 2004) and estimates of ARLs using the Markov approach were computed with the MATLAB technical computing package (The Mathworks, 2004).

For the plots in Column 1, where rounding was used to compute transition probabilities, the ARL estimations vary erratically as the number of Markov states vary. In Plots A1 and B1 the ARLs estimated using the Markov approach are within the 95% confidence limits after 2,520 and 2,830 discretizations of $(0, h)$, but, in Plots C1 and D1, there are
estimates outside the confidence limits as the number of Markov states approach 3,000. On the other hand, for Plots A2, B2, C2 and D2 where smoothing was used to calculate the transition probabilities, the estimates of the ARLs cross the lower 95% confidence bound for 150, 320, 570, and 940 Markov states, respectively, and remain within the confidence limits as the number of discretizations increase.

The degree of instability of the ARL approximations, which were calculated using rounded transition probabilities, appears to increase as the magnitude of the ARL increases. Table 1 shows that the maximum relative differences between the simulated ARL of each of the four CUSUM schemes in Study 1 and each of the equivalent ARLs approximated using the rounding approach increases from 1.12% to 9.68% as the “true” ARL increases.

The Parsonnet score (Parsonnet et al., 1989) and the EuroSCORE (Nashef et al., 1999) are two discrete risk scores used to predict the probability of mortality of patients undergoing cardiac surgical operations. In Study 2, we consider CUSUM schemes, risk-adjusted using these scores, to monitor the number of deaths following cardiac surgery. We approximate ARLs using the Markov approach and compare the estimates of the ARL found using rounding with those found using smoothing.

Figures 2(a) and (b) give the distributions of patient populations categorized by the Parsonnet score into 48 discrete levels of risk (Crayford, 2000) and the EuroSCORE into 20 levels of risk (Bridgewater et al., 2003), respectively. The probability of mortality for each patient category was found using Equation (3) where, for the Parsonnet score, the parameter \((\alpha, \beta)^T\) was taken as \((-3.68, 0.077)^T\) given by Steiner et al. (2000) and, for the EuroSCORE, as \((-5.56, 0.340)^T\) estimated using the data provided by Bridgewater et al. (2003) to fit a simple logistic regression model with the EuroSCORE as the explanatory variable and the mortality outcome as the response.

The plots in Figure 3 are as described for Figure 1 except that, for the plots in Row A, the monitoring scheme is a CUSUM risk-adjusted by the Parsonnet score and with the decision threshold \(h\) set at 4.5, and, for Row B, it is a CUSUM risk-adjusted by the EuroSCORE and with \(h\) at 3.5. In Column 1, where the transition probabilities are found using rounding,
the ARLs of risk-adjusted CUSUMs vary unpredictably as the number of discretizations vary. Clearly, the instability of the ARLs of the CUSUM schemes risk-adjusted by cardiac surgical scores continues after 3,000 discretizations and it is more pronounced than any seen in Column 1 of Figure 1. Table 1 shows that the maximum relative differences between ARLs estimated using simulation and those approximated using the Markov approach are higher than any found for the CUSUM schemes in Study 1.

The values of the ARLs, approximated using the Markov approach with smoothing to calculate the transition probabilities, remain stable as the number of discretizations of \((0, h)\) vary. For risk-adjustment using the Parsonnet score, the approximate ARLs lie within the 95% confidence interval benchmark after 520 discretizations of \((0, h)\) (Figure 3 A2) and, for the EuroSCORE, after 450 discretizations (Figure 3 B2).

4. DISCUSSION

In the two studies in Section 3, we compared two methods of calculating transition probabilities where the Markov approach is used to approximate ARLs of CUSUM schemes risk-adjusted using discrete, finite, risk models.

For transition probabilities calculated using rounding, we found the ARL approximations vary unpredictably as the number of divisions of the decision interval \((0, h)\) into discrete Markov states varies. The degree of instability decreases as the number of discretizations increases, but there are two other factors which may influence this instability. From Study 1, it is clear that there is more pronounced instability of the ARLs estimated using the Markov approach as the “true” value of the ARL of interest increases. A third factor is the distribution of the patient populations. The true values of ARLs for the CUSUM schemes risk adjusted by Parsonnet score and EuroSCORE and the scheme with uniform risk adjustment and \(h\) set to 5.5 are close to 8,000, but the distributions of the patient populations being monitored are different. The ARLs of the CUSUM risk-adjusted by EuroSCORE show greatest instability, and the ARLs of the scheme risk-adjusted by Parsonnet score show greater instability than the ARLs of the CUSUM with uniformly distributed risk-adjustment. We conclude there are at least three factors correlated with the degree of instability in the approximated ARLs;
they are the number of discretizations of the decision interval, the magnitude of the ARL of the risk-adjusted CUSUM scheme, and distribution of the patient population categorized by risk score.

Where smoothing was used to compute the transition probabilities, the approximate ARLs of all the risk-adjusted CUSUM schemes studied converge smoothly from below to values that lie within the 95% confidence bounds for the ARLs estimated using simulation. For each scheme, less than 1,000 discretizations were required for the Markov approximation to be within the confidence interval benchmark.

Approximation of the ARLs of any CUSUM scheme using the Markov approach requires some numerical procedure to solve Equation (4). Relative confidence in the numerical solution depends on the conditioning of the matrix \((I - R)\) (Burden and Faires, 1997). We found that \((I - R)\) was ill-conditioned if either rounding or smoothing was used to compute transition probabilities. Despite the poor conditioning, MATLAB, which embeds “state of the art software for matrix computation” (MATLAB, 2004), provided stable solutions that were consistent with the ARLs found using simulation, if we calculated transition probabilities using smoothing. We assume, therefore, MATLAB also provides numerically stable solutions to Equation (4) if rounding is used to compute transition probabilities and, consequently, it is the method of calculating transition probabilities which causes instability in the estimates of the ARLs.

Although this instability decreases as the number of Markov states increases, the accuracy of any one solution is uncertain. For example, for the CUSUM scheme risk-adjusted by EuroSCORE, we found that the ARL approximation with 2,580 discretizations of \((0, 3.5)\) exceeded the simulation estimate by 10.6%. On the other hand, the smoothing method outlined in Section 2.2 consistently provided good ARL approximations with less than 1,000 discretizations. It should be used to find transition probabilities if the Markov approach is used to estimate ARLs of CUSUM schemes risk-adjusted with discrete, finite risk-models.

5. ACKNOWLEDGMENTS

We would like to thank the Australian Research Council and the Princess Alexandra
Hospital, Brisbane for providing funding for this research through an ARC Hospital Linkage grant, Dr Anthony Morton and Dr David Cook for their advice on the use of control charts for quality control in medicine, and the reviewer for helpful suggestions for presenting the ideas in this paper.

BIBLIOGRAPHY


Table 1: Maximum relative differences between ARL approximations found using simulation and those found using the Markov approach with transition probabilities calculated using rounding.

<table>
<thead>
<tr>
<th>Distribution and Decision Threshold</th>
<th>Markov States†</th>
<th>ARL Estimate Simulated</th>
<th>Markov</th>
<th>Relative Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform*, $h = 4.5$</td>
<td>590</td>
<td>93.9</td>
<td>92.9</td>
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<td>Uniform, $h = 3.5$</td>
<td>1,000</td>
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<td>Uniform, $h = 4.5$</td>
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<tr>
<td>Uniform, $h = 5.5$</td>
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<td>7,192.8</td>
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</tr>
<tr>
<td>Parsonnet, $h = 4.5$</td>
<td>850</td>
<td>8,782.7</td>
<td>10,108.0</td>
<td>13.76%</td>
</tr>
<tr>
<td>EuroSCORE, $h = 3.5$</td>
<td>510</td>
<td>8,112.1</td>
<td>11,781.0</td>
<td>45.23%</td>
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</tbody>
</table>

†Number of discretizations $\geq 500$ for maximum relative difference

*Out-of-control process
Figure 1: Study 1. The solid lines show the plot of approximate ARLs found using the Markov approach versus the number of discretizations of the decision interval \((0, h)\). The dashed and dotted lines in Columns 1 and 2, respectively, give the upper and lower 95% confidence limits of the approximate ARLs found using simulation.
Figure 2: Cardiac surgical populations. Classified by (a) Parsonnet score, taken from Crayford (2000) and (b) EuroSCORE, taken from Bridgewater et al. (2003).
Figure 3: Study 2. The solid lines show the plot of approximate ARLs found using the Markov approach versus the number of discretizations of the decision interval $(0, h)$. The dotted lines give the upper and lower 95% confidence limits of the approximate ARLs found using simulation.