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[Bruza, P.D.](#), [Iqbal, A.](#), & [Kitto, K.](#) (2010) *The role of non-factorizability in determining "pseudo-classical" non-separability*. In: Quantum Informatics for Cognitive, Social, and Semantic Processes (QI 2010), 11-13 November 2010, Arlington, Virginia.

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# The Role of Non-factorizability in Determining “Pseudo-classical” Non-separability

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## Abstract

This article introduces a “pseudo classical” notion of modelling non-separability. This form of non-separability can be viewed as lying between separability and quantum-like non-separability. Non-separability is formalized in terms of the non-factorizability of the underlying joint probability distribution. A decision criterium for determining the non-factorizability of the joint distribution is related to determining the rank of a matrix as well as another approach based on the chi-square-goodness-of-fit test. This pseudo-classical notion of non-separability is discussed in terms of quantum games and concept combinations in human cognition.

Keywords: quantum games, quantum cognition

## Introduction

This article explores how factorizability is used as a formal means for expressing non-separability in quantum, or quantum-like systems. Perhaps the most famous example is John Bell’s factorization of the joint probability distribution with respect to a hidden random variable  $\lambda$  (with prior distribution  $\rho(\lambda)$ ) across two detectors A, B in an EPR setting (Bell 1987):

$$\Pr(A, B) = \int \Pr(A|\lambda) \Pr(B|\lambda) \rho(\lambda) d\lambda$$

The RHS of this equation was an attempt to equate locality with separability expressed as a factorization of the joint probability distribution. What it actually expresses, or may express, has been subject to quite some debate (Dickson 1998). It is not our intention in this article to adjudicate the many subtleties in these debates. Rather, our intention is pragmatically straightforward. One of the aspirations of the quantum interaction community is to model macro phenomena in a quantum-like way. One of the striking features of quantum systems is entanglement. Broadly speaking, entanglement suggests that a system is non-separable. By this we mean, that it cannot be satisfactorily modelled by reducing the system into component parts, whereby the whole is simply assumed to a “sum of

the parts”. Most, if not all, modelling approaches ultimately have this reductive nature. The question then arises whether entanglement can be used to model quantum-like non-separability of macro phenomena. This question has quite some significance when one considers that reductive approaches continually fail to produce satisfactory models of contextual (e.g., complex) systems. Even though the question is highly speculative, there have been several attempts to model the non-separability of macro phenomena by means of entanglement, for example, (Aerts *et al.* 2000; Bruza *et al.* 2009; Aerts, Czachor, & D’Hooghe 2005; Gabora & Aerts 2009). The formal device for determining non-separability has been the Bell inequalities, as is the case in quantum physics. We note in passing that a substantial portion of this work is investigating the non-separability of cognitive phenomena, e.g., words in human memory. For this reason, the term “non-separability” is more apt than the usual term employed in physics “non-locality” as the space separation of the systems is not a relevant feature in cognition. This does not mean we can simply assume non-locality to be a manifestation of the non-separability in space separated systems - the distinction between these two concepts is subtle and not at all clear.

Lurking in the background of some of the work just mentioned is the assumption that non-separability is equivalent to entanglement. The present article challenges this assumption by providing examples of systems that are non-separable, but not entangled. In addition, formal methods are presented for detecting non-separability. These methods essentially rest on the premise that non-separability equates with the non-factorizability, with the aforementioned caveat that such non-separability need not imply entanglement.

## Non-factorizability and non-separability

Broadly speaking, non-separability can be formalized in terms of factorization of a joint probability distribution, or factorization of Hilbert spaces (e.g., (Eakins & Jaroszkiewicz 2003)). This article will focus on probabilistic approaches.

## Probabilistic approaches to non-separability

A probabilistic model of some phenomena requires the identification of a suitable set of random variables. These define

a joint probability distribution. How does one determine whether the system just modelled is non-separable?

Zanotti and Suppes (Suppes & Zanotti 1981) proved a theorem which we use as starting point to explore this question.

**Theorem 1 (Suppes–Zanotti)** *Let  $X_1, \dots, X_n$  be a two-valued random variable. Then a necessary and sufficient condition that there is a random variable  $\lambda$  such that  $X_1, \dots, X_n$  are conditionally independent given  $\lambda$  is that there exists a joint probability distribution of  $X_1, \dots, X_n$*

Note the strength of this theorem - it rests on a logical equivalence. What this theorem entails is that as soon as there is joint probability distribution, there exists a random variable  $\lambda$  which factorizes it:  $\Pr(X_1, \dots, X_n) = \Pr(X_1|\lambda) \dots \Pr(X_n|\lambda)$ . Conversely, if no such hidden variable exists, the joint probability distribution doesn't exist either.

At first sight, this theorem would seem to present a major obstacle to using standard probability theory to model non-separability. The theorem tells us that even if we were tempted to confound non-separability with interconnectness, we can't, because the most deeply interconnected Bayesian network can simply be reduced to a product of its component variables. We take this factorization as an expression of separability in the style of Bell shown above (Aspect, Dalibard, & Roger 1982).

It is important to understand that the random variable  $\lambda$  used to allow the factorization to go through is highly contrived. On this point, Suppes and Zanotti state, "... but it is important to emphasize that the artificial character of  $\lambda$  severely limits its scientific interest". Despite the strength of the theorem its relevance to day-to-day modelling is questionable because in many situations we are interested in whether a *particular* variable may render the model separable, rather than the question of whether such a variable exists. The following section demonstrates such a model.

### The non-separability of bi-ambiguous concept combinations

There is a growing body of literature describing quantum-like models of the conceptual level of cognition, for example, (Bruza & Cole 2005; Aerts & Gabora 2005; Gabora, Rosch, & Aerts 2008; Bruza, Widdows, & Woods 2009; Bruza *et al.* 2009). In recent work, we have speculated that bi-ambiguous concept combinations may behave like entangled bi-partite quantum systems (Bruza *et al.* 2010) Bi-ambiguous concept combinations have the property that both words in a combination are ambiguous. For example, consider the combination "boxer bat". Both words have an animal and sport sense. Our concern in this article, however, is to show how non-separability may be explored in terms of factorizing the associated joint probability distribution.

A bi-ambiguous concept combination of two words is modelled by two random variables  $A$  and  $B$ , where  $A$  corresponds to the first word in the combination and  $B$  corresponds to the second word.

The variable  $A$  ranges over  $\{a_1, a_2\}$  corresponding to its two underlying senses, whereby  $a_1$  is used to refer to

	$b_1$	$b_2$
$a_1$	$p_1$	$p_2$
$a_2$	$p_3$	$p_4$

Table 1: "True" joint distribution

	$b_1$	$b_2$
$a_1$	$q_1$	$q_2$
$a_2$	$q_3$	$q_4$

Table 2: Joint distribution assuming separability

the dominant sense of first word in the combination and  $a_2$  refers to its subordinate sense. Similarly  $B$  ranges over  $\{b_1, b_2\}$ . This convention helps explain the model to follow but is not necessary for its probabilistic development.

In a web-based experiment (Bruza *et al.* 2010), primes are used to orient the subject to towards a certain sense in relation to one of the words. For example, the word "vampire" is used to prime the animal sense of "bat". Thereafter, the concept combination "boxer bat" is presented, and the subjects asked to interpret it, e.g., "a furry black animal with boxing gloves on". In a subsequent step, the subject is asked how they interpreted the sense of each word, e.g. sport for "boxer" and animal for "bat". Primes are designed to span four mutually exclusive cases. By way of illustration, the primes used for "boxer bat" are {fighter, dog, vampire, ball}. The primes are modelled as a random variable  $\lambda$  ranging over  $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ . In a probabilistic setting, separability is formalized by assuming the joint probability is factorizable:

$$\Pr(A, B|\lambda) = \Pr(A|\lambda) \Pr(B|\lambda) \quad (1)$$

Using Bayes' rule, this can be rewritten as:

$$\Pr(A, B, \lambda) = \Pr(A|\lambda) \Pr(B|\lambda) \Pr(\lambda) \quad (2)$$

Assuming the law of total probability:

$$\Pr(A, B) = \sum_{1 \leq i \leq 4} \Pr(A|\lambda_i) \Pr(B|\lambda_i) \Pr(\lambda_i) \quad (3)$$

The final equation allows us to test the separability assumption expressed by the factorization of the joint distribution over the senses of the words (Equation 1). A group of subjects aren't primed, but are simply presented with the concept combination and asked to interpret it. That is, the "true" joint probability distribution is determined empirically and has the general form depicted in Table 1 whereby  $\Pr(A = a_1, B = b_1) = p_1$  denotes the probability the first word  $A$  in the combination is interpreted in its dominant sense ( $A = a_1$ ). Similarly for the second word  $B$ . Each priming condition can be modelled by a marginal distribution  $\Pr(A, B|\lambda_i), 1 \leq i \leq 4$ . These data allow the joint probability distribution  $\Pr(A, B)$  to be computed assuming separability (equation 3) as well as assuming uniform prior probabilities of the primes. This produces a distribution with the general form shown in Table 3. The question then is whether the base joint distribution (see table 1)

and that computed under the assumption of separability (see table 3) are really different. To address this question, a chi-square goodness-of-fit test can be employed, e.g., at the 95% confidence level. This method was employed over a number of bi-ambiguous concept combinations with the results being inconclusive due to lack of statistical power (Bruza *et al.* 2010).

The main point of presenting this analysis is to introduce the possibility that the joint probability distribution is non-factorizable, and hence represents a model which is non-separable and yet the system is not necessarily quantum entangled. We can see this by applying Suppes and Zannotti's theorem - entanglement equates with the non-existence of the joint probability distribution, whereas in this case the joint distribution does exist as manifested by the "true" distribution.

A few remarks are warranted about the general applicability of the above method for establishing the presence of non-separability in a probabilistic model. First, a "true" joint probability distribution is required. If the theoretic distribution is unknown, it can be empirically estimated. Further, the analysis rests on three assumptions. The first is a uniform distribution is assumed for the prior probabilities for the variable  $\lambda$ . In the the case of priming words, a uniform distribution was assumed. This turns out to be a justifiable assumption as the frequency of the prime does not have an effect in extra-list human memory experiments (Bruza *et al.* 2009). In other applications, the distribution across the priors will have to be suitably justified.

The second assumption is the law of total probability holds:  $Pr(A, B) = \sum_i Pr(A, B|\lambda_i) Pr(\lambda_i)$ . The break down of this assumption is core to the field of quantum interaction (Pothos & Bussemeyer 2009; Khrennikov 2010). The LHS can be taken to represent the "true" joint distribution, and RHS is computed from each marginal distribution  $Pr(A, B|\lambda_i)$ , for each  $\lambda_i$ . Together with the priors  $Pr(\lambda_i)$ , the RHS can be computed. The question is whether the LHS=RHS? Once again, chi-square-goodness-of-fit test can be applied for this purpose. If the assumption of total probability holds, then the only remaining assumption is the separability assumption (Equation 3). If the joint probability distribution computed using this equation is deemed not to be the same as the "true" joint probability distribution via the chi-square-goodness-of-fit test, then the culprit is the separability assumption. It can therefore be concluded that the probabilistic model is non-separable as the joint distribution is not factorizable.

### The non-separability of quantum games

In the area of quantum games one proposed approach (Iqbal & Cheon 2007; Iqbal & Abbot 2009) investigates the relationship between new solutions and outcomes in a quantum game in relation to the non-factorizability of an underlying set of probabilities. Such a set can be associated to a quantum system, which the players share in order to physically implement a quantum game. Non-factorizable joint probabilities leading to new outcome(s) in quantum games have been reported in (Iqbal & Cheon 2007; Iqbal & Abbot 2009) when factorizable probabilities correspond to the classical

outcome(s). The other approach (Iqbal & Abbot 2010) considers the situation when a joint distribution does not exist, or cannot be defined, whose marginals constitute a given set of probabilities—in terms of which the players' payoff relations are expressed. Without using the mathematical machinery of quantum mechanics, both these two approaches motivate developing an analysis of quantum games directly from sets of peculiar quantum mechanical joint probabilities.

Consider a two-player two-strategy ( $2 \times 2$ ) game in which Alice's pure strategies are  $S_1$  and  $S_2$  while Bob's pure strategies are  $S'_1$  and  $S'_2$ . A well known example of such a game is the Matching Pennies (MP) (Binmore 2007; Rasmusen 2001) game in which players Alice and Bob each have a penny and each player secretly flips his/her penny to head ( $\mathcal{H}$ ) or tail ( $\mathcal{T}$ ) state. No communication takes place between the players and after making their moves they simultaneously return their pennies to a referee. If pennies match referee takes one dollar from Bob and gives it to Alice (payoff +1 for Alice and -1 for Bob). If the pennies do not match, the referee takes one dollar from Alice and gives it to Bob (payoff -1 for Alice, +1 for Bob). This game is usually represented by the payoff matrix  $\mathcal{M}$

$$\mathcal{M} = \begin{array}{c} \text{Alice} \\ \mathcal{H} \\ \mathcal{T} \end{array} \begin{array}{cc} \text{Bob} \\ \mathcal{H} & \mathcal{T} \\ \left( \begin{array}{cc} (+1, -1) & (-1, +1) \\ (-1, +1) & (+1, -1) \end{array} \right), \end{array} \quad (4)$$

which is a bi-matrix consisting of two matrices

$$\mathcal{A} = \begin{array}{c} \mathcal{H} \\ \mathcal{T} \end{array} \begin{array}{cc} \mathcal{H} & \mathcal{T} \\ \left( \begin{array}{cc} +1 & -1 \\ -1 & +1 \end{array} \right), \end{array} \quad \mathcal{B} = \begin{array}{c} \mathcal{H} \\ \mathcal{T} \end{array} \begin{array}{cc} \mathcal{H} & \mathcal{T} \\ \left( \begin{array}{cc} -1 & +1 \\ +1 & -1 \end{array} \right) \quad (5)$$

that specify Alice's and Bob's payoffs, respectively.

In a mixed-strategy game one has the strategy vectors  $\mathbf{x} = (x, 1-x)^T$  and  $\mathbf{y} = (y, 1-y)^T$ , where  $T$  denotes transpose and  $x, y \in [0, 1]$  give the probabilities for Alice and Bob to choose  $S_1$  and  $S'_1$  respectively. This allows us to construct the payoff relations  $\Pi_{A,B}(x, y) = \mathbf{x}^T(\mathcal{A}, \mathcal{B})\mathbf{y}$  where subscripts  $A$  and  $B$  refer to Alice and Bob, respectively.

As defined above, the MP game is played using only two coins. However, one can also find an arrangement that permits playing this game using four biased coins. Here the referee has 4 biased coins and Alice's coins are labelled  $S_1, S_2$  while Bob's coins are  $S'_1, S'_2$ . In a single run each player has to choose one coin out of the two. So that the chosen pair is one of  $(S_1, S'_1), (S_1, S'_2), (S_2, S'_1), (S_2, S'_2)$ . Players return the two chosen coins to the referee and s/he tosses the two coins together and records the outcome. The referee then collects 4 coins (2 tossed and 2 untossed) & prepares them for the next run. Players' payoff relations can now be defined by making the association  $\mathcal{H} \sim +1$  &  $\mathcal{T} \sim -1$  and from the 16 probabilities.

The biases of the four coins can be described by numbers  $r, s, r', s' \in [0, 1]$  giving us the probabilities of four coins  $S_1, S_2, S'_1, S'_2$  to be in the head state, respectively, i.e.  $r = \Pr(S_1 = +1), s = \Pr(S_2 = +1), r' = \Pr(S'_1 = +1),$  and

$s' = \Pr(S'_2 = +1)$ . To consider the joint probabilities we construct the table (6) below,

$$\begin{array}{c} \text{Alice} \\ S_1 \\ S_2 \end{array} \begin{array}{c} +1 \\ -1 \\ +1 \\ -1 \end{array} \left( \begin{array}{cc|cc} \text{Bob} & S'_1 & S'_2 & \\ & +1 & -1 & +1 & -1 \\ p_1 & p_2 & p_5 & p_6 \\ p_3 & p_4 & p_7 & p_8 \\ \hline p_9 & p_{10} & p_{13} & p_{14} \\ p_{11} & p_{12} & p_{15} & p_{16} \end{array} \right), \quad (6)$$

in which  $p_i$  ( $1 \leq i \leq 16$ ). For instance,  $p_4 = \Pr(S_1 = -1; S'_1 = -1)$  and  $p_{14} = \Pr(S_2 = +1; S'_2 = -1)$  etc. Probabilities  $p_i$  ( $1 \leq i \leq 16$ ) are *factorizable* in that they can be obtained from biases  $r, s, r', s' \in [0, 1]$  as  $p_1 = rr'$ ,  $p_5 = rs'$ ,  $p_9 = sr'$ ,  $p_{13} = ss'$ ,  $p_2 = r(1 - r')$ ,  $p_6 = r(1 - s')$ ,  $p_{10} = s(1 - r')$ ,  $p_{14} = s(1 - s')$ ,  $p_3 = r'(1 - r)$ ,  $p_7 = s'(1 - r)$ ,  $p_{11} = r'(1 - s)$ ,  $p_{15} = s'(1 - s)$ ,  $p_4 = (1 - r)(1 - r')$ ,  $p_8 = (1 - r)(1 - s')$ ,  $p_{12} = (1 - s)(1 - r')$ , and  $p_{16} = (1 - s)(1 - s')$ . When one cannot find such  $r, s, r', s' \in [0, 1]$  in terms of which  $p_i$  ( $1 \leq i \leq 16$ ) can be expressed as above then we say that the set  $p_i$  ( $1 \leq i \leq 16$ ) is non-factorizable.

The players' payoff relations can now be defined by introducing four bias vectors  $\mathfrak{r} = (r, 1 - r)^T$ ,  $\mathfrak{s} = (s, 1 - s)^T$ ,  $\mathfrak{r}' = (r', 1 - r')^T$  and  $\mathfrak{s}' = (s', 1 - s')^T$ . In terms of these the 'pure strategy' payoff relations can be written as  $\Pi_{A,B}(S_1, S'_1) = \mathfrak{r}^T(\mathcal{A}, \mathcal{B})\mathfrak{r}'$ ,  $\Pi_{A,B}(S_1, S'_2) = \mathfrak{r}^T(\mathcal{A}, \mathcal{B})\mathfrak{s}'$ ,  $\Pi_{A,B}(S_2, S'_1) = \mathfrak{s}^T(\mathcal{A}, \mathcal{B})\mathfrak{r}'$ , and  $\Pi_{A,B}(S_2, S'_2) = \mathfrak{s}^T(\mathcal{A}, \mathcal{B})\mathfrak{s}'$ . The mixed-strategy payoffs are then obtained from the relation

$$\Pi_{A,B}(x, y) = \mathbf{x}^T \begin{pmatrix} \Pi_{A,B}(S_1, S'_1) & \Pi_{A,B}(S_1, S'_2) \\ \Pi_{A,B}(S_2, S'_1) & \Pi_{A,B}(S_2, S'_2) \end{pmatrix} \mathbf{y}, \quad (7)$$

where  $x$  and  $y$  are the probabilities with which Alice and Bob select the coins  $S_1$  and  $S'_1$ , respectively and

$$\begin{aligned} \Pi_{A,B}(S_1, S'_1) &= \mathfrak{r}^T(\mathcal{A}, \mathcal{B})\mathfrak{r}', & \Pi_{A,B}(S_1, S'_2) &= \mathfrak{r}^T(\mathcal{A}, \mathcal{B})\mathfrak{s}', \\ \Pi_{A,B}(S_2, S'_1) &= \mathfrak{s}^T(\mathcal{A}, \mathcal{B})\mathfrak{r}', & \Pi_{A,B}(S_2, S'_2) &= \mathfrak{s}^T(\mathcal{A}, \mathcal{B})\mathfrak{s}'. \end{aligned} \quad (8)$$

For an arbitrary strategy pair  $(x^*, y^*)$  the Nash inequalities read

$$\begin{aligned} 4(r - s) \{y^*(r' - s') + s' - 1/2\} (x^* - x) &\geq 0, \\ -4(r' - s') \{x^*(r - s) + s - 1/2\} (y^* - y) &\geq 0. \end{aligned} \quad (9)$$

We notice that if  $r + s = 1$  and  $r' + s' = 1$  then  $(x^*, y^*) = (1/2, 1/2)$ ,  $\Pi_A(1/2, 1/2) = 0 = \Pi_B(1/2, 1/2)$ . This re-expresses the Matching Pennies game in terms of 16 factorizable probabilities while giving identical to the classical outcome of the game. Constraints  $r + s = 1$ ,  $r' + s' = 1$  on the four biases ensure that the classical outcome of the game emerges for factorizable probabilities. The definition

of factorizability, described after the table (6), permits  $p_i$  to be written in terms of  $r, s, r', s' \in [0, 1]$  as

$$r = p_1 + p_2, \quad s = p_9 + p_{10}, \quad r' = p_1 + p_3, \quad s' = p_5 + p_7, \quad (10)$$

and the constraints  $r + s = 1$ ,  $r' + s' = 1$  are then expressed as

$$p_1 + p_3 + p_5 + p_7 = 1, \quad p_1 + p_2 + p_9 + p_{10} = 1. \quad (11)$$

A route to obtaining a quantum game consists of retaining these constraints and allowing joint probabilities  $p_i$  to become non-factorizable. That is, we consider the situation that the constraints given in Eqs. (11) hold while one cannot find  $r, s, r', s' \in [0, 1]$  in terms of which  $p_i$  ( $1 \leq i \leq 16$ ) can be expressed as described after the table (6). Note that the constraints (11) ensure that the classical outcome of the game emerges when probabilities become factorizable.

## Discussion

We propose that the basic structure for analyzing non-separability emerges from the above, whether it be games or concept combinations. This structure is depicted in equation 12. The symbols  $A$  and  $B$  can refer to either players or concepts. Similarly, the symbols  $a, b, a', b'$  refer to game strategies, or primes for concepts. The values represent outcomes: payoffs, or dominant (+1), or secondary (-1) senses of words.

$$\begin{array}{c} \text{A} \\ a \\ b \end{array} \begin{array}{c} +1 \\ -1 \\ +1 \\ -1 \end{array} \begin{array}{c} \text{B} \\ a' \\ b' \end{array} \begin{array}{c} +1 \\ -1 \\ +1 \\ -1 \end{array} \left( \begin{array}{cc|cc} p_1 & p_2 & p_5 & p_6 \\ p_3 & p_4 & p_7 & p_8 \\ \hline p_9 & p_{10} & p_{13} & p_{14} \\ p_{11} & p_{12} & p_{15} & p_{16} \end{array} \right), \quad (12)$$

This structure provides a practical means for understanding non-factorizability. If the matrix in equation 12, denoted  $P$  has rank 1, then the corresponding joint probability distribution is factorizable. This is because when the matrix has a rank of 1, then there exist column vectors  $\mathbf{p}, \mathbf{q}$  such that  $\mathbf{p}\mathbf{q}^T = M$ :

$$\begin{aligned} \mathbf{p}^T &= (p_1, p_2, p_3, p_4) \\ \mathbf{q}^T &= (q_1, q_2, q_3, q_4) \end{aligned}$$

As seen above, in quantum Matching Pennies games  $\mathbf{p}, \mathbf{q}$  have the form:  $p_1 = r, p_2 = 1 - r, p_3 = s, p_4 = (1 - s)$  and  $q_1 = r', q_2 = (1 - r'), q_3 = s', q_4 = (1 - s')$ . Quite surprisingly, this is also the case with conceptual combinations where the primes can be considered as equivalent to the strategies  $S_1$  and  $S_2$  and the concept combinations are somewhat like biased pennies. This is because words often have a dominant sense. For example, the dominant sense of "boxer" is the sporting sense, not the animal sense (the breed of dog). Similarly, the dominant sense of "bat" is the animal

sense over the sport sense. The primes can be equated to strategies as follows:

$$\begin{aligned} \text{boxer}^T &= S_1(\text{prime=fighter}) : r, (1-r), \\ &S_2(\text{prime=dog}) : s, (1-s) \\ \text{bat}^T &= S'_1(\text{prime=vampire}) : r', (1-r'), \\ &S'_2(\text{prime=ball}) : s', (1-s') \end{aligned}$$

The primes can be viewed as a sort of strategy to align the human subject towards (or away) from a predisposed sense of a word. The close similarity between the structure of quantum games and conceptual combinations opens the door to model concept combinations in terms of quantum game theory. More research is needed to determine whether this line of thought will bear fruit.

When  $P$  does not have a rank equal to 1, then the joint distribution depicted in equation 12 cannot be factorized into two separate probability distributions  $\mathbf{p}, \mathbf{q}$ . This is a sufficient but not necessary condition to determine non-factorizability of the joint distribution. The joint distribution is factorizable if and only if  $M$  has rank 1 and  $\mathbf{p}^T, \mathbf{q}^T$  have the form  $(r, 1-r, s, 1-s)$  and  $(r', 1-r', s', 1-s')$  respectively. The advantage of translating the problem of determining whether the joint distribution is factorizable or not into one of determining a matrix's rank is that there are direct and efficient linear algebraic means to do this. For example, compute the singular-value decomposition of  $M$  and the rank is equivalent to the number of singular values produced in the decomposition.

### Summary

This article introduces a “pseudo classical” notion that can be used in the modelling non-separability of different phenomena. This form of non-separability can be viewed as lying between separability and quantum-like non-separability. In a sense, quantum-like non-separability can be seen as an “extreme” form of non-separability. Non-separability is formalized in terms of the non-factorizability of the underlying joint probability distribution. A decision criterion for determining the non-factorizability of the joint distribution is related to determining the rank of a matrix as well as another approach based on the chi-square-goodness-of-fit test. This pseudo-classical notion of non-separability was discussed in terms of quantum games and concept combinations in human cognition. The hope is that the proposed method is general enough that it can be for analyzing non-separability in a variety of settings.

It is known that although a non-factorizable set of probabilities may not violate Bell's inequality, a set of probabilities that violates Bell's inequality must be non-factorizable (Winsberg & Fine 2003). For the playing of games this leads to the interesting situation of a game with non-factorizable set of probabilities that does not violate Bell's inequality. Such games have been identified as residing in the so-called “pseudo-classical” domain (Cheon & Tsutsui 2006). They define this domain as being the one where Bell's inequality is not violated, and where a quantum game can be treated as if players are simultaneously playing several classical games. In the case concept combinations,

“pseudo-classical” non-separability undermines reductive models which understand concept combinations solely in terms of the constituent words in the combination. Such reductive models explicitly or tacitly adhere to compositional semantics. We don't argue that the principle of compositionality is wrong, but rather a better understanding is required of when it can be legitimately applied. Whilst the field of quantum information science does not distinguish “pseudo-classical” non-separability, we feel the distinction is a useful one in order to classify quantum-like systems.

### Acknowledgements

This article was supported in part by the Australian Research Council Discovery grants DP0773341 and DP1094974, and by the UK Engineering and Physical Sciences Research Council, grant number: EP/F014708/2.

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