THE RIESZ-BESSEL FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. This paper examines the properties of a fractional diffusion equation defined by the composition of the inverses of the Riesz potential and the Bessel potential. The first part determines the conditions under which the Green function of this equation is the transition probability density function of a Lévy motion. This Lévy motion is obtained by the subordination of Brownian motion, and the Lévy representation of the subordinator is determined. The second part studies the semigroup formed by the Green function of the fractional diffusion equation. Applications of these results to certain evolution equations is considered. Some results on the numerical solution of the fractional diffusion equation are also provided.

1. Introduction

In the Eulerian theory of turbulence, the concentration field $c(t,x)$ is commonly assumed to satisfy the advection-diffusion equation

$$\frac{\partial c}{\partial t} + \nabla.(uc) = \kappa \Delta c(t,x), \quad t \in \mathbb{R}_+,\, x \in D \subset \mathbb{R}^n, \tag{1.1}$$

where ∇ is the gradient vector, $u(t,x)$ is the velocity vector field, κ is the molecular diffusivity coefficient and Δ is the Laplacian. On the other hand, in the Lagrangian theory, the assumptions of model (1.1) are equivalent to assuming that the particle trajectories $x(t)$ satisfy the Itô stochastic differential equation

$$dx(t) = u(t, x(t))\, dt + (2\kappa)^{1/2}\, dB(t), \tag{1.2}$$

where the components of $B(t)$ are independent Brownian motions (Schuss [31], Thomson [33]). In the context of anomalous diffusion, the equivalence between the Eulerian and Lagrangian approaches has not been established, a possible exception being in fractional-in-space diffusion, in which case the particles would follow a Lévy motion. This paper presents a further investigation on this topic.

A fundamental example of fractional-in-space diffusion is the equation

$$\frac{\partial c}{\partial t} = -\kappa(-\Delta)^{\alpha} c(t,x),\ \alpha > 0, \tag{1.3}$$

where $(-\Delta)^{\alpha}$ is understood as the inverse of the Riesz potential defined by the kernel

$$J_{\alpha}(x) = \frac{\Gamma(n/2-\alpha)}{\pi^{n/2} 4^{\alpha} \Gamma(\alpha)} |x|^{2\alpha - n} \tag{1.4}$$

(Stein [32]). For $\alpha \in (0,1]$ the Green function of (1.3) is the symmetric $2\alpha-$stable probability density function and the corresponding Lagrangian theory would assume that the

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particle trajectories follow symmetric $2\alpha-$stable Lévy motions (see Bochner [5], also Feller [12] for an extension to the asymmetric case). In fact, Feller [12] considered the problem of generating all the stable probability distributions through the semigroup (known as the Feller semigroup) generated by the Green function of the diffusion

$$\frac{\partial c}{\partial t} = \mathcal{D}_{\theta}^{2\alpha} c\left(t, x\right), \ t > 0, \ x \in \mathbb{R}, \tag{1.5}$$

where $\mathcal{D}_{\theta}^{2\alpha}$ is the pseudo-differential operator with symbol $\widehat{\mathcal{D}_{\theta}^{2\alpha}} = -\left|\lambda\right|^{2\alpha} \exp\left(i \operatorname{sign}\left(\lambda\right) \theta\pi/2\right)$, α being the index of stability, θ the index of skewness (asymmetry). When $\theta = 0$, (1.5) is reduced to (1.3) with $\kappa = 1$, and the Feller semigroup represents all 2α-stable distributions.

Instead of (1.3), we shall consider the following fractional diffusion equation which allows different behaviours on macro and micro scales:

$$\frac{\partial c}{\partial t} = -\kappa \left(-\Delta\right)^{\alpha} \left(I - \Delta\right)^{\gamma} c\left(t, x\right), \ t > 0, \ x \in \mathbb{R}^n, \tag{1.6}$$

where $\left(I - \Delta\right)^{\gamma}$ is the inverse of the Bessel potential defined by the kernel

$$I_{\gamma}\left(x\right) = \left[\left(4\pi\right)^{\gamma} \Gamma\left(\gamma\right)\right]^{-1} \int_0^{\infty} e^{-\pi|x|^2/s} e^{-s/4\pi} s^{(-n/2+\gamma)} \frac{ds}{s} \tag{1.7}$$

(Stein [32]). The spatial Fourier tranform of the Green function to (1.6) is

$$\widehat{G}\left(t, z\right) = \exp\left[-\kappa t \left|z\right|^{2\alpha} \left(1 + \left|z\right|^2\right)^{\gamma}\right], \ z \in \mathbb{R}^n. \tag{1.8}$$

It should be noted that putting $\gamma = 0$ reduces (1.6) to (1.3). The component $\left(I - \Delta\right)^{\gamma}$ is needed for (1.6) to have finite second-order moments in a non-Gaussian scenario.

In this paper we shall determine the conditions under which (1.8) is the characteristic function of a type G distribution (defined below) and study the properties of the resulting Lévy motion, which will be referred to as the Riesz-Bessel-Lévy motion (RBLm) due to the specific appearance of the Riesz and Bessel operators in (1.6). *This will allow a Lagrangian interpretation to (1.6) in the sense that the particles will follow a Lévy motion with characteristic function (1.8).* The specific form of the characteristic function leads to a simple method for the statistical estimation of RBLm. These results are given in Section 3, after some preliminaries collected in Section 2.

Section 4 will provide full details on the corresponding Lévy measure and self-decomposability of the Riesz-Bessel-Lévy subordinator (RBLs) and motion. It will be seen that while the exponent of the inverse of the Riesz potential indicates how often large jumps occur, it is the combined effect of the inverses of the Riesz and Bessel potentials that describes the multifractal behaviour (Jaffard [18]) of the process. More precisely, depending on the sum of the exponents of the inverses of the Riesz and Bessel potentials, the Lévy motion will be either a compound Poisson process, a pure jump process with jumping times dense in $[0, \infty)$ or the sum of a compound Poisson process and an independent Brownian motion. Thus the two-parameter model (1.6) is able to generate a range of behaviours for the concentration field and of the representative particles.

Section 3 establishes that the Green function of (1.6) is the transition probability density function of a Lévy motion for $0 < \alpha \leq 1$ and $0 \leq \alpha + \gamma \leq 1$, hence forms a strongly continuous semigroup for this range of parameters. Based on the sharper results of Section 4, we show in Section 5 that the Green function of (1.6) in fact defines a strongly continuous semigroup for a wider range of parameters, and determine the form of the infinitesimal generator of this semigroup. The result allows us to consider in Section 6 certain evolution equations defined by this semigroup. Some previous results of Grecksch and Tudor [16], and Grecksch and Anh [14] can be brought into this context to obtain existence, uniqueness and approximation of the solutions of these evolution equations.

Section 7 discusses some numerical methods to simulate RBLs and RBLm. An example will be given. However, the impetus of this section will be a new method to approximate the solution of the fractional diffusion equation (1.6) with Dirichlet boundary conditions. Standard iterative methods, which are commonly shown to be convergent in a Markov setting, do not seem applicable for (1.6) for spatial dimension $n \geq 2$. The investigation of Section 3 yields that the semigroup of this equation can be obtained by subordination of the semigroup from the classical diffusion equation for a range of parameters under the same boundary conditions. The solution to (1.6) is then written in terms of the semigroup of the classical diffusion equation. This leads to a convergent scheme to approximate (1.6).

Finally Section 8 provides some remarks with regard to some extension of the results of this paper to the fractional-in-space and in-time diffusion equation.

2. Preliminaries

2.1. **Subordination of semigroups and stochastic processes.** The concept of infinite divisible, stable and self-decomposable distributions and their characterisations are detailed in Sato [28], [29], for example. In particular, a distribution μ on $\mathbb{R}^n$ is said to be *infinitely divisible* if, for every positive integer k, there is a distribution μ_k such that $\widehat{\mu}(z) = (\widehat{\mu}_k(z))^k$, where $\widehat{\mu}(z) = \int_{\mathbb{R}^n} e^{i\langle x,z\rangle}\mu(dx)$ is the characteristic function of μ. It is said to be *stable* if, for every positive integer k, there are $a > 0$ and $b \in \mathbb{R}^n$ such that $(\widehat{\mu}(z))^k = \widehat{\mu}(az)\, e^{i\langle b,z\rangle}$. A distribution μ is said to be *self-decomposable* if, for every $a > 1$, there is a distribution ρ on $\mathbb{R}^n$ such that $\widehat{\mu}(z) = \widehat{\mu}(a^{-1}z)\,\widehat{\rho}(z)$. Stable distributions are self-decomposable, while self-decomposable distributions are infinitely divisible. Exponential, two-sided exponential and $\Gamma-$distributions are self-decomposable but are not stable. Stable distributions and self-decomposable distributions play an important role in the theory of limit distributions for sums of independent variables (Sato [29]).

A stochastic process $\{X(t), t \geq 0\}$ with values in $\mathbb{R}^n$ is called a *Lévy process* if it is stochastically continuous with stationary independent increments and $X(0) = 0$. Stable processes and self-decomposable processes form important subclasses of Lévy processes. Suppose that $X(t)$ is a Lévy process corresponding to an infinitely divisible distribution μ on $\mathbb{R}^n$. Define the transition probability function

$$P(t, x, B) = \mu^t(B - x),$$

where B is a Borel set, μ^t is the probability measure corresponding to the distinguished t-th power $(\widehat{\mu}(z))^t = e^{t\log\widehat{\mu}(z)}$ of the characteristic function $\widehat{\mu}(z)$ (Sato [28], p. 10). Define for $f \in C_0$

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(y)\, P(t, x, dy) = \int_{\mathbb{R}^n} f(x+y)\, \mu^t(dy).$$

Then the family of operators $\{P_t,\, t \geq 0\}$ is a *strongly continuous semigroup* on $C_0(\mathbb{R}^n)$ with norm $\|P_t\| = 1$ (Sato [28], Section 2 of Chapter 3).

A process $\{Z(t), t \geq 0\}$ is a *subordinator* if it is a non-decreasing Lévy process. Subordination is then a transformation of a semigroup (resp. stochastic process) to a new semigroup (resp. new stochastic process) through random time change by a subordinator. In fact, let $\{Z(t), t \geq 0\}$ be a subordinator with $P_{Z(1)} = \lambda$. Here, as in [28], we write $P_Z(B)$ for $P(Z \in B)$. Let $\{P_t, t \geq 0\}$ be a strongly continuous semigroup of linear contractions on a Banach space $\mathcal{B}$ with infinitesimal generator L. Define

$$Q_t f = \int_{[0,\infty)} P_s f \lambda^t(ds), \quad f \in \mathcal{B}.$$

Then $\{Q_t, t \geq 0\}$ is a strongly continuous semigroup of linear contractions on $\mathcal{B}$. Let M be its infinitesimal operator. The transformation from P_t to Q_t is called *Bochner's subordination* by the subordinator $Z(t)$. The semigroup Q_t and its generator M are said to be *subordinate* to P_t and L respectively.

On the other hand, let $X(t)$ be a Lévy process and $Z(t)$ a subordinator. Suppose that $X(t)$ and $Z(t)$ are independent. Define

$$Y(t) = X(Z(t)), \quad t \geq 0.$$

Then $Y(t)$ is a Lévy process and the transformation from $X(t)$ to $Y(t)$ is a subordination by the subordinator $Z(t)$. Any Lévy process identical in law with $Y(t)$ is said to be subordinate to $X(t)$ (Sato [28], p. 85).

2.2. **Bounded holomorphic semigroups.** Suppose that $\{e^{-tA}, t \geq 0\}$ is a strongly continuous semigroup of operators generated by $-A$. It is known that, for the class of Bernstein functions f, the operator $-f(A)$ is again a generator, that is, $\{e^{-tf(A)}, t \geq 0\}$ is a strongly continuous semigroup of operators. Bernstein functions are those for which $x \longrightarrow e^{-tf(x)}$ is completely monotone $\forall\, t \geq 0$. The class of Bernstein functions is denoted by $\mathbb{B}$. Berg *et al.* [4] extended this result to the case of $-A$ generating a bounded strongly continuous holomorphic semigroup of angle θ. We outline some of their results here as these are needed in Section 5.

For $0 < \psi \leq \pi/2$, put $S_\psi = \{re^{i\phi}; r > 0, |\phi| < \psi\}$. The strongly continuous semigroup $\{e^{-tA}, t \geq 0\}$ is a *bounded holomorphic semigroup of angle* ψ it it extends to a holomorphic family of bounded operators $\{e^{-zA}, z \in S_\psi\}$ that is bounded and strongly continuous in $\overline{S_\phi}$ whenever $0 < \phi < \psi$.

Suppose $0 \leq \theta < \pi/2$. Then A is of *type* θ if $-A$ generates a bounded holomorphic semigroup of angle $\pi/2 - \theta$ (Berg *et al.* [4], Theorem 3.4).

(2.1) *If A is an operator of type θ, $n \in \mathbb{N}$ and $n\theta < \pi/2$, then A^n is of type $n\theta$.*

(2.2) *Suppose that A is of type* 0 *and $f \in \mathbb{B}$. Then $f(A)$ is of type* 0

(Berg *et al.* [4], Theorem 7.2). In particular, let $A = -\Delta$, the negative Laplacian on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then $f(A)$ is of type 0 $\forall f \in \mathbb{B}$. It is noted that the class $\mathbb{B}$ includes fractional powers $z \longrightarrow z^\alpha$, $0 < \alpha < 1$, $z \longrightarrow \log(1+z)$, $z \longrightarrow \sqrt{z}\, \arctan(1/\sqrt{z})$.

2.3. **Type-G distributions.** The class of type G distributions was introduced in Marcus [21] and the class of processes with type G finite-dimensional distributions was further studied by Rosinski [27]. A random variable X is said to be of type G if $X = \sigma\varepsilon$ where equality is in distribution, $\sigma > 0$ and ε are independent random variables with σ^2 being infinitely divisible and ε having a standard normal distribution. A multivariate extension was proposed by Barndorff-Nielsen and Pérez-Abreu [2] and Maejima and Rosinski [22]. The multivariate Student distribution, hyperbolic distributions, symmetric α-stable distributions, and symmetric normal inverse Gaussian distributions are among special cases of type-G distributions (Barndorff-Nielsen and Pérez-Abreu [2]). Standard conditioning arguments show that if

$$E\left(e^{-\sigma^2 u}\right) = \exp\left(-g\left(u\right)\right), \tag{2.3}$$

then

$$E\left(e^{i\langle X,u\rangle}\right) = \exp\left(-g\left(\frac{u^T\Sigma^{-1}u}{2}\right)\right). \tag{2.4}$$

3. The Riesz-Bessel distribution

We first establish that the Green function of the fractional diffusion equation (1.6) is the density function of a type-G distribution under certain conditions.

Theorem 1. *The function $\widehat{G}\left(t,z\right)$ of (1.8) is the characteristic function of a type G distribution for all $t \geq 0$ if and only if $\alpha \in \left(0,1\right], \alpha+\gamma \in \left[0,1\right]$.*

Proof. We first show that the function

$$f\left(x\right) = x^{\alpha-1}\left(1+x\right)^{\gamma-1}\left(\alpha+\left(\alpha+\gamma\right)x\right) \tag{3.1}$$

is completely monotonic for all $x > 0$. Applying the product rule we have

(3.2)

$$f^{(n)}\left(x\right) = x^{\alpha-n-1}\left(1+x\right)^{\gamma-n-1}\left\{P_n\left(x\right)\left(\left(\alpha-n\right)+\left(\alpha+\gamma-2n\right)x\right)+P_n'\left(x\right)x\left(1+x\right)\right\},$$

where $P_n\left(x\right)$ is a polynomial in x of order n. Let C_n^m denote the coefficient of x^m in the $n-$th polynomial; then the following recursive relationships follow directly from (3.2):

$$C_{n+1}^0 = \left(\alpha-n\right)C_n^0, \tag{3.3}$$

$$C_{n+1}^m = \left(\alpha+m-n\right)C_n^m + \left(\alpha+\gamma+m-2n-1\right)C_n^{m-1}, \tag{3.4}$$

$$C_{n+1}^{n+1} = \left(\alpha+\gamma-n\right)C_n^n \tag{3.5}$$

with $C_1^0 = \alpha$ and $C_1^1 = \alpha+\gamma$. By mathematical induction, the following recurrence relationship holds:

$$C_{n+1}^n = C_n^{n-1}\frac{n+1}{n}\left(\alpha+\gamma-n\right). \tag{3.6}$$

>From (3.3) - (3.6) it follows that all non-zero coefficients of $P_n\left(x\right)$ are of the sign $\left(-1\right)^{n+1}$ if $\alpha+\gamma \in \left[0,1\right]$ and $\alpha \in \left(0,1\right]$. Hence, $f\left(x\right)$ is a completely monotonic function for $x > 0$. Applying Theorem 1 of Feller [13, Chapter XIII-7] we have that

$$\exp\left[-\kappa t z^{\alpha}\left(1+z\right)^{\gamma}\right] \tag{3.7}$$

is the Laplace transform of an infinitely divisible probability measure on $\left[0,\infty\right)$. It then follows from (2.4) that $\widehat{G}\left(t,z\right)$ is the characteristic function of a type G random variable.

To prove the only if part of the theorem we suppose that $\widehat{G}\left(t,z\right)$ is a characteristic function for $\alpha+\gamma > 1$ and let $\varkappa = \left(2\alpha+2\gamma\right)^{-1}$. It follows that $\widehat{G}\left(ht,h^{-\varkappa}z\right)$ is a characteristic function for all $h > 0$ and

$$\begin{aligned}
\widehat{G}\left(ht,h^{-\varkappa}z\right) &= \exp\left(-\kappa t h^{1-2\alpha\varkappa}\left|z\right|^{2\alpha}\left(1+h^{-2\varkappa}\left|z\right|^2\right)^{\gamma}\right) \\
&= \exp\left(-\kappa t\left|z\right|^{2\alpha}\left(h^{2\varkappa}+\left|z\right|^2\right)^{\gamma}\right) \\
&\to \exp\left(-\kappa t\left|z\right|^{2(\alpha+\gamma)}\right) = \psi\left(z\right),
\end{aligned}$$

where convergence is pointwise on $\mathbb{R}^n$ as $h \to 0$. As $\psi(z)$ is continuous at $z = 0$ for $\alpha + \gamma > 0$ it follows from the continuity theorem that it is a characteristic function; but this leads to a contradiction since it is known that $\psi(z)$ is a characteristic function only for $\alpha + \gamma \in (0, 1]$. By letting $\varkappa = 1/2\alpha$ and $h \to \infty$ in $\widehat{G}(ht, h^{-\varkappa}z)$ it follows by similar arguments that α must be less than or equal to one. If $\alpha \leq 0$ then $\widehat{G}(t, 0) \neq 1$ and hence is not a characteristic function. If $\alpha + \gamma < 0$ then $\widehat{G}(t, z) \to 1$ as $|z| \to \infty$, which implies that $\widehat{G}(t, z) \equiv 1$ and hence a contradiction. ∎

In the above theorem there are two types of distributions. The first has characteristic function (1.8) and the resulting Lévy motion will be referred to as the Riesz-Bessel-Lévy motion. The second has Laplace transform (3.7) and the resulting Lévy motion will be referred to as the Riesz-Bessel-Lévy subordinator. In each case the parameters are the triple (κ, α, γ). The relationship between the RBLm, RBLs and the symmetric stable motions and subordinators is summarised in the following corollary to Theorem 1. The proof of Corollary 1 is straightforward from the proof of Theorem 1.

Corollary 1. *Let $X(t)$ be a RBLm and $Z(t)$ a RBLs each with parameters (κ, α, γ). The following properties hold:*

- *As $t \to \infty$, $t^{-1/2\alpha}X(t)$ (resp. $t^{-1/\alpha}Z(t)$) converges in distribution to a symmetric $2\alpha-$stable random variable (resp. positive $\alpha-$stable random variable).*
- *Assume that $\alpha + \gamma > 0$, then as $t \to 0$, $t^{-1/2(\alpha+\gamma)}X(t)$ (resp. $t^{-1/(\alpha+\gamma)}Z(t)$) converges in distribution to a symmetric $2(\alpha + \gamma)-$stable random variable (resp. postive $(\alpha + \gamma)-$stable random variable).*
- *Let $S(t)$ be a $\beta-$stable subordinator with Laplace transform*

$$E\left(e^{-zS(t)}\right) = \exp\left(-t|z|^{\beta}\right), \tag{3.8}$$

then $X(S(t))$ is RBLm (resp. $Z(S(t))$ is RBLs) with parameters $\left(\kappa^{\beta}, \alpha\beta, \gamma\beta\right)$.

The above results explain why we refer to (1.6) as having different behaviours on the micro ($t \to 0$) and macro ($t \to \infty$) scales.

3.1. **Parameter estimation.** Parameter estimation of the Riesz-Bessel distribution given an independent and identically distributed sequence of random variables can be carried out using the sample characteristic function defined by

$$\widehat{\phi}(\lambda) = \frac{1}{N}\sum_{k=1}^{N} \exp(i\lambda x_k), \tag{3.9}$$

which is known to be a strongly consistent estimator. Various methods for estimating the parameters of the stable distribution have been based on $\hat{\phi}$ (Press [25], Paulson *et al.* [23], Koutrovelis [19], [20]). Simulations suggest that the regression method of Koutrovelis is most efficient and so we propose a similar approach for parameter estimation of the Riesz-Bessel distribution. The following expression follows directly from (1.8):

$$\log\left(-\log\left|\widehat{G}(t, z)\right|^2\right) = \log(2\kappa t) + 2\alpha\log|z| + \gamma\log\left(1 + |z|^2\right). \tag{3.10}$$

Estimates of κ, α and γ can be obtained by replacing $\widehat{G}(t, z)$ by its sample estimate and using linear regression on $\log|z|$ and $\log\left(1 + |z|^2\right)$. In this setting, the problem of choosing between a stable distribution and a Riesz-Bessel distribution is simply one of model selection in linear regressions.

4. Lévy representation

4.1. **Lévy measure.** It is known that the Laplace transform of an infinitely divisible distribution on $[0,\infty)$ can be written as

$$\log E\left(e^{-yX}\right) = -cy - \int_0^\infty \left(1-e^{-\lambda y}\right) P\left(d\lambda\right), \tag{4.1}$$

where P is the Lévy measure of the distribution. The Lévy measure of the RBLs can be written explicitly in terms of Kummers confluent hypergeometric function ${}_1F_1$ for all values of α and γ. Recall that ${}_1F_1$ is an entire function with series expansion

$${}_1F_1\left(a;b;x\right) := \sum_{k=0}^{\infty} \frac{x^m}{m!}\frac{(a)_m}{(b)_m},$$

where $(a)_n$ denotes the shifted factorial

$$(a)_n = a\left(a+1\right)\ldots\left(a+n-1\right).$$

We note that from Bernstein's theorem (Feller [13, Theorem 1a of Chapter XIII-5]) there exists a positive measure $Q\left(d\lambda\right)$ on $[0,\infty)$ so that the function $f\left(x\right)$ of (3.1) can be written as

$$f\left(x\right) = \int_0^\infty e^{-\lambda x} Q\left(d\lambda\right). \tag{4.2}$$

If $\alpha+\gamma\in[0,1)$, then $f\left(x\right)\to 0$ as $x\to\infty$ and hence Q does not possess an atom at zero. From a Tauberian theorem (Feller [13, Theorem 2 of Chapter XIII-4]) the growth rate on Q can be bounded so that in integrating f from 0 to y we may interchange the order of integration to get

$$y^\alpha\left(1+y\right)^\gamma = \int_0^\infty \left(1-e^{-\lambda y}\right)\lambda^{-1} Q\left(d\lambda\right). \tag{4.3}$$

It follows from (4.3) that *the Lévy measure of the RBLs is $\kappa\lambda^{-1}$ times the inverse Laplace transform of $f\left(x\right)$*. For $\alpha+\gamma\in[0,1)$ we may use the tables of inverse Laplace transforms (Prudnikov *et al.* [26, Eq. 2.1.2-1]) to obtain

$$\begin{aligned} P\left(d\lambda\right) &= \kappa\left(\frac{\alpha\lambda^{-\alpha-\gamma}}{\Gamma\left(2-\alpha-\gamma\right)}\,{}_1F_1\left(1-\gamma;2-\alpha-\gamma;-\lambda\right)\right. \\ &\quad \left. + \frac{\left(\alpha+\gamma\right)\lambda^{-\alpha-\gamma-1}}{\Gamma\left(1-\alpha-\gamma\right)}\,{}_1F_1\left(1-\gamma;1-\alpha-\gamma;-\lambda\right)\right)d\lambda. \end{aligned} \tag{4.4}$$

As ${}_1F_1$ is a smooth function it follows that *for $\alpha+\gamma\in(0,1)$ the Lévy measure is not a finite measure and hence the RBLs is not a compound Poisson process.* The special cases $\alpha=1$ and $\alpha+\gamma=0$ yield particularly simple expressions for the Lévy measure. Substituting $\alpha=1$ into (4.4) and applying the series representation of ${}_1F_1$ we obtain the simplification

$$P\left(d\lambda\right) = \frac{\kappa}{\Gamma\left(-\gamma\right)}\left(\lambda^{-(1+\gamma)}e^{-\lambda} + \left(1+\gamma\right)\lambda^{-(2+\gamma)}e^{-\lambda}\right)d\lambda. \tag{4.5}$$

It follows that *RBLs with $\alpha=1$ is the sum of a compound Poisson process with Gamma distributed jumps and an exponentially tempered stable subordinator* (see Barndorff-Nielsen and Shepard [3]). Substituting $\alpha+\gamma=0$ in (4.4) gives

$$P\left(d\lambda\right) = \alpha\kappa\,{}_1F_1\left(1+\alpha,2,-\lambda\right)d\lambda \tag{4.6}$$

which, from the integral representation of ${}_1F_1$, is a member of the class of generalised convolutions of mixtures of exponential distributions (Bondesson [6], [7]). *RBLs with* $\alpha+\gamma=0$ *is therefore a compound Poisson process.*

The case of $\alpha+\gamma=1$ needs special attention because $f(x)\to 1$ as $x\to\infty$ and hence Q in (4.2) has an atom at zero. Let $\widetilde{Q}(d\lambda)=Q(d\lambda)-\delta_0(\lambda)$ then $\widetilde{Q}$ has no atom at zero and

$$y^{\alpha}(1+y)^{\gamma}-y=\int_0^{\infty}\left(1-e^{-\lambda y}\right)\lambda^{-1}\widetilde{Q}(d\lambda). \tag{4.7}$$

To determine the Lévy measure a series expansion of $f(x)-1$ is required:

$$\begin{aligned} x^{\alpha-1}(1+x)^{-\alpha}(\alpha+x)-1 &= \alpha x^{-1}\left(1+x^{-1}\right)^{-\alpha}+\left(1+x^{-1}\right)^{-\alpha}-1 \\ &= \sum_{k=0}^{\infty}\left(\alpha c(k,-\alpha)+c(k+1,-\alpha)\right)x^{-(k+1)}, \end{aligned}$$

where $c(k,-\alpha)$ are the binomial coefficients and the series converges absolutely for $x>1$. Term-by-term inversion of the Laplace transform is valid by Theorem 35.1 of Doetsch [11] and hence the Lévy measure is

$$P(d\lambda)=\kappa t\left(\sum_{k=0}^{\infty}\frac{\lambda^{k-1}\left(c(k+1,-\alpha)+\alpha c(k,-\alpha)\right)}{k!}\right)d\lambda. \tag{4.8}$$

A simplification in terms of ${}_1F_1$ is possible by noting that

$$c(k,-\alpha)=\frac{(-1)^k(\alpha)_k}{(1)_k}$$

and by the series representation of ${}_1F_1$ we have

$$\begin{aligned} P(d\lambda) &= \kappa t\left(\alpha\lambda^{-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(\alpha)_k}{(1)_k}-\lambda^{-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(\alpha)_{k+1}}{(1)_{k+1}}\right)d\lambda \\ &= \alpha\kappa t\lambda^{-1}\left({}_1F_1(\alpha;1;-\lambda)-{}_1F_1(\alpha+1;2;-\lambda)\right)d\lambda. \qquad (4.9) \end{aligned}$$

It follows from the properties of ${}_1F_1$ that this measure is finite and so *the RBLs is a compound Poisson process with drift* κt *for* $\alpha+\gamma=1$.

4.2. **Self-decomposability of the Riesz-Bessel-Lévy motion.** From the Lévy representation of a subordinator on $\mathbb{R}$, the concept of self-decomposability has the following characterisation (see Sato [28, Corollary to Theorem 6.1]):

Theorem 2. *A subordinator is self-decomposable if and only if*

$$E\left(e^{-zS_t}\right)=\exp\left(-ctz+t\int_0^{\infty}\left(e^{-\lambda z}-1\right)\frac{k(\lambda)}{\lambda}d\lambda\right) \tag{4.10}$$

with $c\geq 0$ *and* $k(\lambda)$ *being a non-negative function, decreasing and right-continuous on* $(0,\infty)$ *and satisfying*

$$\int_0^{\infty}(1\wedge\lambda)\frac{k(\lambda)}{\lambda}d\lambda<\infty.$$

The function $k(\lambda)$ is called the $k-$function of the one-dimensional self-decomposable distribution.

Theorem 3. *For the Riesz-Bessel-Lévy subordinator the following properties hold:*
(i) If $\alpha+\gamma=0$ or $\alpha+\gamma=1$, then it is not self-decomposable.
(ii) If $\alpha=1$, then it is self-decomposable if and only if $\gamma>-3/4$.
(iii) If $\gamma\leq-4\alpha^2/(1+4\alpha)$, then it is not self-decomposable.

Proof. For $\alpha+\gamma=0$ and $\alpha+\gamma=1$ the Lévy measure of the RBLs is a finite measure and hence is not self-decomposable. If $\alpha=1$ then the k-function is

$$k(\lambda)=\frac{\kappa}{\Gamma(-\gamma)}\left(\lambda^{-\gamma}e^{-\lambda}+(1+\gamma)\lambda^{-(1+\gamma)}e^{-\lambda}\right)$$

and differentiating with respect to λ yields

$$k'(\lambda)=\frac{-\kappa\lambda^{-(2+\gamma)}e^{-\lambda}}{\Gamma(-\gamma)}\left(\lambda^2+(1+2\gamma)\lambda+(1+\gamma)^2\right). \tag{4.11}$$

It is seen that $k(\lambda)$ is decreasing if and only if

$$4(1+\gamma)^2-(1+2\gamma)^2>0, \tag{4.12}$$

and hence the RBLs is self-decomposable if and only if $\gamma>-3/4$. In the general case, the Laplace transform of the k-function is $f(x)$ given in (3.1). From the basic properties of the Laplace transform

$$\int_0^\infty e^{-\lambda x}(\lambda k'(\lambda))\,d\lambda=-(f(x)+xf'(x)), \tag{4.13}$$

it follows from Bernstein's theorem and Theorem 2 that the RBLs is self-decomposable if and only if the function

$$x^{\alpha-1}(1+x)^{\gamma-2}\left(\alpha^2+(2\alpha(\alpha+\gamma)+\gamma)x+(\alpha+\gamma)^2x^2\right) \tag{4.14}$$

is completely monotonic. Applying the quadratic formula, (4.14) is non-positive for $x>0$ if $\gamma<0$ and

$$\left(2\alpha(\alpha+\gamma)^2+\gamma\right)^2-4\alpha^2(\alpha+\gamma)^2\geq0. \tag{4.15}$$

Hence, the RBLs is not self-decomposable if $\gamma\leq-4\alpha^2/(1+4\alpha)$. ∎

Remark 1. *We can use the fact that RBLs is self-decomposable if and only if (4.14) is completely monotone to determine numerically a region of the parameter space which results in a self-decomposable distribution. We can avoid performing numerical differentiation by applying Theorem 11d of Widder* [35, Chapter IV] *which states that if $f(x)$ is a completely monotonic function on $[a,\infty)$ and if δ is any positive number, then the sequence $\{f(a+n\delta)\}_{n=0}^\infty$ is completely monotonic, that is,*

$$(-1)^k\Delta^kf(a+n\delta)\geq0,\quad\forall k,n=0,1,2,\ldots \tag{4.16}$$

Therefore, if we find values of α,γ and δ which do not result in a completely monotonic sequence, then we can state that for the pair (α,γ) RBLs is not self-decomposable. If the sequence formed is completely monotonic, then this provides evidence in favour of RBLs being self-decomposable but is not a proof. Figure 1 was constructed by taking $k=0,1,\ldots,10,a=0.01,\delta=0.01$ and $n=5000$. In the figure the outer lines define the region of parameters for RBLs, that is $\alpha+\gamma\in[0,1]$. The regions between the curves and straight lines give the range of parameter values for which RBLs is not self-decomposable. Note that the region on the left of the graph has a similar shape to that given in part *(iii) of Theorem 4. Also, the largest value of γ which results in a RBLs not being self-decomposable when $\alpha = 1$ was numerically determined to be -0.77, which is close to the value determined analytically in part (ii) of Theorem 3. On the other hand, the region on the right of the graph was only determined numerically and we have no analytical results for this range of parameter values.*

The above properties of the RBLs can be used to study the RBLm. As RBLm is subordinated Brownian motion, it follows that the Lévy measure of RBLm is

$$\nu(d\lambda) = \int_0^\infty (4\pi s)^{-n/2} \exp\left(\frac{-|\lambda|^2}{4s}\right) P(ds)\, d\lambda, \tag{4.17}$$

where $P(ds)$ is the Lévy measure of the subordinator (see Theorem 7.2 of Sato[28]). When $\alpha + \gamma < 1$, RBLm has no Brownian component; however for $\alpha + \gamma = 1$ there is a Brownian component with covariance matrix $2\kappa I$. From the form of RBLm with $\alpha + \gamma = 1$ and the subordination property, it follows that for $\gamma > 0$ the Lévy measure has the decomposition

$$\nu(d\lambda) = \frac{C}{|\lambda|^{1+2(\alpha+\gamma)}} d\lambda + \tilde{\nu}(d\lambda), \tag{4.18}$$

where $\tilde{\nu}(d\lambda)$ is a finite Lévy measure. Halgreen [17] proved that subordinated Brownian motion is self-decomposable if the subordinator is self-decomposable. However, it is not necessary that the subordinator be self-decomposable for the resulting Lévy motion to be self-decomposable (Sato [30]).

Corollary 2. *For the Riesz-Bessel-Lévy motion the following properties hold:*
(i) If $\alpha + \gamma = 0$ or $\alpha + \gamma = 1$, then it is not self-decomposable.
(ii) If $\alpha = 1$ and $\gamma > -3/4$, then it is self-decomposable.

5. The Riesz-Bessel semigroup

In Section 3, the conditions under which the Green function of (1.6) is the transition probability density function of a Lévy motion were determined. Many of the properties of this distribution can be given in terms of semigroup properties on the Banach space $(C_0(\mathbb{R}^n), \sup|\cdot|)$. In particular,

- As the Green function of (1.6) is the transistion probability density function of a Lévy motion for $\alpha \in (0,1]$, $\alpha+\gamma \in [0,1]$, it also forms a strongly continuous contraction semigroup for this range of parameters. Furthermore, as this Lévy motion is a subordinated Brownian motion, this semigroup is obtained by subodination, in the sense of Bochner, of the semigroup formed by the classical diffusion equation.
- For $\alpha + \gamma = 0$, the resulting RBLm is a compound Poisson process. The resulting semigroup must therefore be uniformly continuous and the infinitesimal generator is a bounded linear operator.
- For $\gamma > 0$, the RBLm can be written as the sum of a symmetric $2(\alpha+\gamma)-$ stable motion and an independent compound Poisson process. It is known that the infinitesimal generator of a semigroup of symmetric stable motion is $-(-\Delta)^{(\alpha+\gamma)}$ and the infinitesimal generator of a compound Poisson process is a bounded linear operator. It follows that the infinitesimal generator of RBLm with $\gamma > 0$ may be written as the closure of

$$-\kappa(-\Delta)^{\alpha+\gamma} + \Phi_{\alpha,\gamma,\kappa}, \tag{5.1}$$

where $\Phi_{\alpha,\gamma,\kappa}$ is a bounded linear operator (see Theorem 1.1 of Chapter 3 in Pazy [24]).

Motivated by these results we aim to show that (i) the Green function of (1.6) defines a strongly continuous semigroup for a wide range of parameters, (ii) when $\alpha+\gamma=0$ the semigroup is uniformly continuous, and (iii) for a range of its parameters the infinitesimal generator of the Riesz-Bessel semigroup has the representation (5.1). In proving these results we shall work in the Banach space $L^p(\mathbb{R}^n)$; however the results also hold for $(C_0(\mathbb{R}^n), \sup|\cdot|)$.

Theorem 4. *The Green function of (1.6) defines a strongly continuous, bounded, holomorphic semigroup of angle* $\pi/2$ *on* $L^p(\mathbb{R}^n)$ *for* $\alpha>0, \alpha+\gamma\geq 0$ *and any* $p\geq 1$.

Proof. Assume $\alpha\in(0,1]$, and $\alpha+\gamma\in[0,1]$. As $x^{\alpha-1}(1+x)^{\gamma-1}(\alpha+(\alpha+\gamma)x)$ is a completely monotonic function (see the proof of Theorem 1), it follows from a theorem of Bochner (see, for example, Berg *et al.* [4, Theorem 2.1]) that $x^{\alpha}(1+x)^{\gamma}$ is a Bernstein function. From Theorem 7.2 of Berg *et al.* [4] (and their Example 7.3, or (2.1) and (2.2) above) it follows that $(-\Delta)^{\alpha}(I-\Delta)^{\gamma}$ is of type 0 on $L^p(\mathbb{R}^n)$, that is, the semigroup generated by $-(-\Delta)^{\alpha}(I-\Delta)^{\gamma}$ is a bounded holomorphic semigroup of angle $\pi/2$. Now let $\alpha>0$, $\gamma>0$ and $\alpha+\gamma\geq 0$. If $n-1<\alpha+\gamma<n$, define $\tilde{\alpha}=\alpha/n$ and $\tilde{\gamma}=\gamma/n$ and hence $(-\Delta)^{\tilde{\alpha}}(I-\Delta)^{\tilde{\gamma}}$ is of type 0. >From a theorem of deLaubenfels [9] $(-\Delta)^{\alpha}(I-\Delta)^{\gamma}$ is also of type 0 (see Theorem 3.5 of Berg *et al.* [4]). Thus $-(-\Delta)^{\alpha}(I-\Delta)^{\gamma}$ is the infinitesimal generator of a bounded holomorphic semigroup of angle $\pi/2$ and the semigroup is given by the Green function of (1.6). Similarly the result holds for $\gamma<0$. ∎

Let M be the smallest integer greater than or equal to $\alpha+\gamma$ and assume that $\alpha+\gamma\notin\mathbb{N}$. Define

$$\begin{aligned} g(x) &= x^{\alpha}(1+x)^{\gamma}-\sum_{m=0}^{M}c_{\gamma}(m)\,x^{\alpha+\gamma-m} \\ &= \sum_{m=M+1}^{\infty}c_{\gamma}(m)\,x^{\alpha+\gamma-m}, \quad |x|>1, \end{aligned}$$

where the infinite sum is absolutely convergent for $|x|>1$. By Theorem 35.2 of Doetsch [11] we may invert the Laplace transform term-by-term, yielding

$$\tilde{g}(s)=\sum_{m=M+1}^{\infty}\frac{c_{\gamma}(m)}{\Gamma(m-\alpha-\gamma)}s^{m-\alpha-\gamma-1}$$

and the sum is convergent for all $s\neq 0$. From Theorem 41.1 of Doetsch [11] an asymptotic expansion for $\tilde{g}(s)$ can be determined from an expansion of g, that is,

$$g(x)=\sum_{m=0}^{\infty}c_{\gamma}(m)\,x^{\alpha+m}-\sum_{m=0}^{M}c_{\gamma}(m)\,x^{\alpha+\gamma-m},$$

where the series converges absolutely in a neighbourhood of $x=0$. The asymptotic expansion for $\tilde{g}$ is then

$$\tilde{g}(s)\sim\sum_{m=0}^{\infty}\frac{c_{\gamma}(m)}{\Gamma(-\alpha-m)}s^{-(\alpha+m+1)}-\sum_{m=0}^{M}\frac{c_{\gamma}(m)}{\Gamma(m-\alpha-\gamma)}s^{m-\alpha-\gamma-1}.$$

Define the function

$$G(x) = \int_0^\infty (4\pi s)^{-n/2} e^{-|x|^2/4s} \tilde{g}(s)\, ds. \tag{5.2}$$

It should be clear that G is bounded for all $x \in \mathbb{R}^n$ and $G(x) \sim C|x|^{-(n+\varkappa)}$ for some finite constant C and $\varkappa = \min(2\alpha, 2(M-\alpha-\gamma))$. If $\alpha+\gamma \in \mathbb{N}$ then define

$$\begin{aligned} g(x) &= x^\alpha (1+x)^\gamma - \sum_{m=0}^{M} c_\gamma(m)\, x^{\alpha+\gamma-m} \\ &= c_\gamma(\alpha+\gamma) + \sum_{m=1}^{\infty} c_\gamma(\alpha+\gamma+m)\, x^{-m}, \quad |x|>1, \end{aligned}$$

where the infinite sum is absolutely convergent for $|x| > 1$. Applying term-by-term inversion of the Laplace transform,

$$\begin{aligned} \tilde{g}(s) &= c_\gamma(\alpha+\gamma)\,\delta_0(s) + \sum_{m=0}^{\infty} \frac{c_\gamma(\alpha+\gamma+1+m)}{m!} s^m \\ &= c_\gamma(\alpha+\gamma)\,\delta_0(s) + \tilde{g}_1(s) \end{aligned}$$

and the sum is convergent for all s. To determine the asymptotic expansion of $\tilde{g}_1$, we note that

$$g_1(x) = \sum_{m=0}^{\infty} c_\gamma(m)\, x^{\alpha+m} - \sum_{m=0}^{M-1} c_\gamma(m)\, x^{\alpha+\gamma-m}$$

which converges absolutely in a neighbourhood of $x = 0$. The asymptotic expansion of $\tilde{g}_1$ is then

$$\tilde{g}_1(s) \sim \sum_{m=0}^{\infty} \frac{c_\gamma(m)}{\Gamma(-\alpha-m)} s^{-(\alpha+m+1)}.$$

Define the functions

$$G_1(x) = \int_0^\infty (4\pi s)^{-n/2} e^{-|x|^2/4s} \tilde{g}_1(s)\, ds$$

and

$$G(x) = c_\gamma(\alpha+\gamma)\,\delta_0(x) + G_1(x).$$

It is clear that $G_1(x)$ is bounded for all $x \in \mathbb{R}^n$ and $G_1(x) \sim C|x|^{-(n+2\alpha)}$ for some finite constant C. Finally, for all $\alpha+\gamma \geq 0$ define the operator $\Phi_{\alpha,\gamma,\kappa}$ by

$$\Phi_{\alpha,\gamma,\kappa}(f) = \kappa G * f$$

for $f \in L^p(\mathbb{R}^n)$. We now show that $\Phi_{\alpha,\gamma,\kappa}$ is a bounded linear operator on $L^p(\mathbb{R}^n)$.

Lemma 1. *$\Phi_{\alpha,\gamma,\kappa}$ is a bounded linear operator on $L^p(\mathbb{R}^n)$.*

Proof. Clearly $\Phi_{\alpha,\gamma,\kappa}$ is a linear operator on $L^p(\mathbb{R}^n)$ and hence it is sufficient to show that it is a bounded operator. Assume $\alpha+\gamma \in \mathbb{N}$ and for simplicity $\kappa = 1$, then

$$\begin{aligned}
|\Phi_{\alpha,\gamma,1}(f)| &\leq |f| + |G_1 * f| \\
&\leq |f| + \int_{\mathbb{R}^n} |G_1(u) f(x-u)|\, du \\
&\leq |f| + \left(\int_{\mathbb{R}^n} |G_1(u)|^{p'} \left(1+|u|^2\right)^{\nu p'/2} du\right)^{1/p'} \qquad (5.3)\\
&\quad \times \left(\int_{\mathbb{R}^n} |f(x-u)|^p \left(1+|u|^2\right)^{-\nu p/2} du\right)^{1/p}
\end{aligned}$$

for any $\nu > 0$ provided the integrals exist, and (5.3) follows from Hölder's inequality. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Phi_{\alpha,\gamma,1}(f)|^p\, dx &\leq 2^p \int_{\mathbb{R}^n} |f(x)|^p\, dx + 2^p \left(\int_{\mathbb{R}^n} |G_1(u)|^{p'} \left(1+|u|^2\right)^{\nu p'/2} du\right)^{p/p'} \\
&\quad \times \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-u)|^p \left(1+|u|^2\right)^{-\nu p/2} du\right) dx \\
&= 2^p \parallel f \parallel_p^p + 2^p \parallel f \parallel_p^p \left(\int_{\mathbb{R}^n} \left(1+|u|^2\right)^{-\nu p/2} du\right) \qquad (5.4)\\
&\quad \times \left(\int_{\mathbb{R}^n} |G_1(u)|^{p'} \left(1+|u|^2\right)^{\nu p'/2} du\right)^{p/p'},
\end{aligned}$$

where the equality in (5.4) follows from an application of Tonelli's theorem. To show that $\Phi_{\alpha,\gamma,1}$ is a bounded operator we need $\nu p > n$ and

$$\int_{|u|>1} |G_1(u)|^{p'} |u|^{\nu p'}\, du < \infty.$$

>From the asymptotic behaviour of G_1 this integral is finite for

$$n/p < \nu < n/p + 2\alpha/p.$$

Finally, as ν is arbitrary, we have established that $\Phi_{\alpha,\gamma,\kappa}$ is a bounded linear operator for $\kappa = 1$ and $\alpha+\gamma \in \mathbb{N}$. It is clear that this is true for general $\kappa > 0$. For $\alpha+\gamma \notin \mathbb{N}$ the proof follows the same lines. ∎

Theorem 5. *(i) If $\alpha+\gamma = 0$ the Riesz-Bessel semigroup is uniformly continuous on $L^p(\mathbb{R}^n)$ with infinitesimal generator $\Phi_{\alpha,\gamma,\kappa}$.*
(ii) If $\alpha+\gamma \in (0,1]$ the infinitesimal generator of the Riesz-Bessel semigroup can be written as the closure of (5.1).

Proof. (i) Consider the Cauchy problem

$$\begin{aligned}
\frac{du(t)}{dt} + \Phi_{\alpha,\gamma,\kappa} u(t) &= 0, \qquad (5.5)\\
u(0) &= f,
\end{aligned}$$

where $f \in L^p(\mathbb{R}^n)$. The solution to (5.5) is given by $S(t) f$, where $S(t)$ is the semigroup generated by $\Phi_{\alpha,\gamma,\kappa}$. Partial Fourier transform of (5.5) gives

$$\begin{aligned} \frac{d\hat{u}(t)}{dt} + \kappa |\lambda|^{2\alpha} \left(1 + |\lambda|^2\right)^{\gamma} \hat{u}(t) &= 0, \\ \hat{u}(0) &= \hat{f} \end{aligned}$$

and hence $S(t)$ is the inverse Fourier transform of $\exp\left(-\kappa t |\lambda|^{2\alpha} \left(1 + |\lambda|^2\right)^{\gamma}\right)$ which is the Riesz-Bessel semigroup. From Lemma 1 $\Phi_{\alpha,\gamma,\kappa}$ is a bounded linear operator and hence $S(t)$ is uniformly continuous.
(ii) It is known that $-\kappa(-\Delta)^{\alpha+\gamma}$ is the infinitesimal generator of a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ and it follows that the closure of (5.1) is also an infinitesimal generator of a strongly continuous semigroup on $L^p(\mathbb{R}^n)$. The resulting semigroup is seen to be the Riesz-Bessel semigroup by applying the same arguments as in the proof of (i). ∎

Remark 2. *In the corollary to Theorem 1 we see that, for t small, RBLm is similar to* $2(\alpha + \gamma)$ *−stable motion. As a consequence of Theorem 5, a similar property is true for the Riesz-Bessel semigroup with* $\alpha + \gamma \in [0, 1]$*. If* $T(t)$ *is the semigroup generated by the closure of* $-\kappa(-\Delta)^{\alpha+\gamma}$ *and* $S(t)$ *is the Riesz-Bessel semigroup, then applying Corollary 1.3 in Chapter 3 of Pazy* [24] *we have*

$$\| S(t) - T(t) \| \leq \exp(\| \Phi_{\alpha,\gamma,\kappa} \| t) - 1.$$

Therefore, for t small, there is little difference between the semigroup generated by $-(-\Delta)^{\alpha}(I - \Delta)^{\gamma}$ *and that generated by* $-(-\Delta)^{\alpha+\gamma}$*.*

6. Numerical Methods

6.1. **Simulation of Riesz-Bessel-Lévy motion.** Simulation of infinitely divisible distributions given their Lévy representations has been studied by a number of authors. One approach is to approximate the random variables by truncating a series representation (e.g. Bondesson [7]), another is to approximate the random variable by a compound Poisson process (e.g. Damien *et al.* [8]). In principle, the simulation of a compound Poisson process on $\mathbb{R}$ with Lévy measure μ is straightforward. One first simulates a Poisson process N_t with rate $\mu\{\mathbb{R}\}$ and then forms the sum

$$X_t = \sum_{m=1}^{N_t} \zeta_i,$$

where ζ_i is a sequence of independent and identically distributed (iid) random variables with distribution $\mu/\mu\{\mathbb{R}\}$. Thus, in principle, it is straightforward to simulate the Riesz-Bessel-Lévy subordinator for $\alpha + \gamma = 1$ and $\alpha + \gamma = 0$. However, for $\alpha + \gamma = 1$ it is difficult to generate values from a distribution proportional to (4.9) and standard methods for simulation such as rejection sampling will require further examination of the Lévy measure. These problems will be investigated elsewhere.

Simulation of the Riesz-Bessel-Lévy subordinator for the special case $\alpha = 1$ is more promising. We have previously noted that the form of the Lévy measure (4.5) implies that this RBLs is the sum of an exponentially tempered stable (ETS) subordinator and an independent compound Poisson process. Rosinski ([27]) proposed a method for simulating random variables from the ETS distribution using a series representation. Rosinski determined that the random variable given by the series

$$\sum_{i=1}^{\infty} \min \left\{ \left(\frac{a_i C}{A} \right)^{-1/C}, e_i v_i^{1/C} \right\}$$

is an ETS random variable with Lévy measure

$$u(x)\, dx = A x^{-C-1} e^{-Bx} dx \quad x > 0.$$

Here, a_i is the arrival times of a Poisson process with intensity 1, e_i is a sequence of iid exponentially distributed random variables with mean $1/B$ and v_i is a sequence of iid standard uniform random variables. Thus, to simulate the RBLs with $\alpha = 1$, we set

$$A = \frac{\kappa(1+\gamma)}{\Gamma(-\gamma)}, \quad B = 1, \quad C = 1 + \gamma, \tag{6.1}$$

and simulate an ETS subordinator and then add an independent compound Poisson process with intensity κ and jumps distributed according to a Gamma distribution with shape parameter $-\gamma$ and scale parameter 1.

The special cases $\alpha + \gamma = 1$ and $\alpha = 1$ form the basis for simulating RBLs and RBLm. From these two cases all others can be obtained by subordination by a $\beta-$ stable subordinator Y_t with Laplace transform

$$E\left(e^{-zY_t}\right) = \exp\left(-tz^{\beta}\right). \tag{6.2}$$

If $\gamma < 0$ then set $\beta = \alpha, \widetilde{\alpha} = 1, \widetilde{\gamma} = \gamma/\beta$ and $\widetilde{\kappa} = \kappa^{1/\beta}$. If $\gamma > 0$ then set $\beta = \alpha + \gamma, \widetilde{\alpha} = \alpha/\beta, \widetilde{\gamma} = \gamma/\beta$ and $\widetilde{\kappa} = \kappa^{1/\beta}$. A simple conditioning argument shows that subordination of RBLs$(\widetilde{\kappa}, \widetilde{\alpha}, \widetilde{\gamma})$ with Y_t equals in distribution to RBLs(κ, α, γ). Thus, in principle, we are able to simulate all Riesz-Bessel-Lévy subordinators. Simulation of RBLm can then be performed by simulating a Brownian motion with covariance matrix $2I$ and subordinating it by an RBLs.

Figure 2 below shows a simulated sample path of RBLm in $\mathbb{R}^2$ with $\alpha = 1$ and $\gamma = -1$. Its two components over time are displayed in Figure 3. The parameter α controls the tail behaviour of the distribution of the Lévy motion while γ (through the value of $\alpha+\gamma$) controls the small scale structure of the Lévy motion. In the above plots we see that the Lévy motion is a compound Poisson process $(\alpha + \gamma = 0)$ as all movements of the particle are through jumps. None of the jumps is very large due to the parameter $\alpha = 1$ and hence the distribution has finite moments of all orders. In Figure 3 we see clearly that jumps in x and y directions occur simultaneously. This is a result of the vector-valued RBLm being obtained by subordination.

6.2. **Approximation of fractional diffusion equation on a bounded domain.** It is known that $-\Delta$ is an operator of type 0 on $L^p(\Omega)$, with Dirichlet boundary conditions, for $1 \leq p < \infty$, where $\Omega \subset \mathbb{R}^n$ is a region with smooth boundary. As given by Theorem 4, the Green function of (1.6) must form a bounded holomorphic semigroup of angle $\pi/2$ on $L^p(\Omega)$, with Dirichelt boundary conditions, for $1 \leq p < \infty$. Futhermore, for $\alpha \in (0,1], \alpha + \gamma \in [0,1]$ the semigroup is obtained by subordination of the semigroup from the classical diffusion equation, under the same boundary conditions. The solution to

$$\frac{\partial c(t,x)}{\partial t} = -\kappa(-\Delta)^{\alpha}(I-\Delta)^{\gamma}c(t,x), \quad x\in\Omega, \tag{6.3}$$

$$c(t,x) = 0, \quad x\in\partial\Omega, \tag{6.4}$$

$$c(0,x) = c_0(x) \tag{6.5}$$

is given by $c(t,x)=S(t)\,c_0(x)$, where $S(t)$ is the semigroup formed by (6.3) and (6.5). The semigroup may be written as

$$S(t)=\int_0^{\infty}H(\sigma)\,\mu_t(d\sigma), \tag{6.6}$$

where $H(t)$ is the semigroup of the classical diffusion equation under the boundary conditions and μ_t is the distribution of the RBLs at time t. The solution to (6.3) - (6.5) may then be written as

$$c(t,x)=\int_0^{\infty}H(\sigma)\,c_0(x)\,\mu_t(d\sigma), \tag{6.7}$$

that is, *the solution to the fractional diffusion equation (1.6) on a smooth bounded domain with Dirichlet boundary conditions can be obtained by averaging the solution to the classical diffusion equation, with the same boundary conditions, over time according to the distribution of the Riesz-Bessel-Lévy subordinator.*

Recall that $H(t)\,c_0(x)$ is continuous in t for all $t\in[0,\infty)$ and differentiable in t for all $t\in(0,\infty)$. Assuming that $\alpha+\gamma>0$, then the measure μ has a density. Let K_N be a sequence of compact sets growing to $(0,\infty)$ as $N\to\infty$. It now follows that

$$\int_{K_N^C}H(\sigma)\,c_0(x)\,\mu(d\sigma)\to 0$$

as $N\to\infty$ and thus we may approximate (6.7) by

$$c(t,x)=\int_{K_N}H(\sigma)\,c_0(x)\,\mu_t(d\sigma).$$

Let Π_N be a partition of K_N such that $|\Pi_N|\to 0$ as $N\to\infty$. Continuity of the solution to the classical diffusion equation means that

$$\left|\int_{K_N}H(\sigma)\,c_0(x)\,\mu_t(d\sigma)-\sum_{\Pi_N}H(\tilde{\sigma}_i)\,c_0(x)\,\mu\{[\sigma_i,\sigma_{i+1})\}\right|\to 0$$

as $N\to\infty$ and under some conditions on K_N and Π_N, where $\sigma_i<\tilde{\sigma}_i<\sigma_{i+1}$. Finally, the classical diffusion equation can be approximated by a finite difference scheme so that

$$\left|H\widehat{(t)\,c_0}(x)-H(t)\,c_0(x)\right|\to 0,$$

where $H\widehat{(t)\,c_0}(x)$ is the approximate solution to the classical diffusion equation. Thus, we propose to approximate (6.7) by

$$\sum_{\Pi_N}H\widehat{(\tilde{\sigma}_i)\,c_0}(x)\mu\{[\sigma_i,\sigma_{i+1})\},$$

which is convergent under certain conditions on K_N, Π_N and the parameters of the finite difference scheme.

7. Stochastic evolution equations

We have seen that the Green function of (1.6) forms a strongly continuous semigroup on the Banach space $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$. This result can be used to study the fractional-in-space stochastic evolution equation on $L^2(\Omega)$:

$$\begin{aligned} dX(t) &= -\kappa(-\Delta)^{\alpha}(I-\Delta)^{\gamma} X(t)\,dt + f(t, X(t))\,dt + g(t, X(t))\,dW(t), \\ X(0) &= X_0, \quad X(t,x)\,|_{x\in\partial\Omega} = 0, \end{aligned} \tag{7.1}$$

with $\alpha > 0, \alpha+\gamma \geq 0, \kappa > 0$, where $W(t)$ is a Hilbert space-valued Wiener process and the stochastic integral is defined in the Itô sense.

Suppose that the Wiener process takes values in the separable Hilbert space K with the covariance operator Q as its kernel operator. Let $\mathcal{L}_Q$ be the Hilbert space of linear operators $B : Q^{1/2}K \mapsto H$ so that $BQ^{1/2}$ is a Hilbert-Schmidt operator from K into H, where H is a separable Hilbert space. The norm in $\mathcal{L}_Q$ is denoted by $\|\cdot\|_Q$. The following assumptions are imposed on the functions f and g of (7.1):

$$\begin{aligned} \| f(t,u) - f(t,v) \|_H + \| g(t,u) - g(t,v) \|_Q &\leq L \| u - v \|_H, \\ \| f(t,u) + g(t,u) \|_H &\leq L(1 + \| u \|_H) \end{aligned}$$

for all $t \in [0,T]$, $u, v \in H$ and L a positive constant. If $E\|X_0\|_H^2 < \infty$, then there exists a pathwise unique solution to (7.1) given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s) f(s, X(s))\,ds + \int_0^t S(t-s) g(s, X(s))\,dW(s), \tag{7.2}$$

where $S(t)$ is the Riesz-Bessel semigroup on $L^2(\Omega)$ (see Theorem 3.1, p. 88 of Grecksch and Tudor [16]).

Following Grecksch and Anh [15], a splitting method described below can be used to obtain a time discretisation of (7.2). The idea is to construct two sequences of equations where the first sequence defines a deterministic problem with probability one, while the second sequence defines a purely stochastic problem. The two sequences are linked through the initial conditions of the deterministic problem.

7.1. Splitting method.

An increment $X(t) - X(r)$, $r < t$, of the solution process $X(t)$ (7.2) can be written as

$$\begin{aligned} X(t) - X(r) &= \int_0^r [S(t-s) f(s, X(s)) - S(r-s) f(s, X(s))]\,ds \\ &\quad + \int_0^t S(t-s) f(s, X(s))\,ds \\ &\quad + \int_0^r [S(t-s) g(s, X(s)) - S(r-s) g(s, X(s))]\,dW(s) \\ &\quad + \int_0^t S(t-s) g(s, X(s))\,dW(s). \end{aligned} \tag{7.3}$$

Eq. (7.3) suggests the following splitting method. Let $0 = t_0^N \leq t_1^N \leq \ldots \leq t_{m_N}^N = t$ be partitions of $[0,t] \subset [0,T]$ with

$$\max\left\{ t_{j+1}^N - t_j^N : 0 \leqq j \leqq m_N - 1 \right\} =: h_N \to 0 \text{ for } N \to \infty.$$

For simplicity, we write t_j for t_j^N. We define the processes $\left(X_{1t}^N(s), X_{2t}^N(s)\right), s \in [0,t]$, by

$$\begin{aligned} X_{1t}^N(s) &= X_{1t}^N(t_k) \\ &\quad + \int_0^{t_k} \left[S(t_{k+1}-u) f\left(u, X_{1t}^N(u)\right) - S(t_k - u) f\left(u, X_{1t}^N(u)\right)\right] du \\ &\quad + \int_{t_k}^{s} S(t_{k+1}-u) f\left(u, X_{1t}^N(u)\right) du \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} X_{2t}^N(s) &= X_{2t}^N(t_k) \\ &\quad + \int_0^{t_k} \left[S(t_{k+1}-u) g\left(u, X_{1t}^N(u)\right) - S(t_k - u) g\left(u, X_{1t}^N(u)\right)\right] dW(u) \\ &\quad + \int_{t_k}^{s} S(t_{k+1}-u) g\left(u, X_{1t}^N(u)\right) dW(u) \end{aligned} \tag{7.5}$$

with

$$X_{1t}^N(t_k) = X_{2t}^N(t_k - 0), \tag{7.6}$$

$$X_{2t}^N(t_k) = X_{1t}^N(t_{k+1} - 0), \tag{7.7}$$

$$X_{1t}^N(0) = X_0, \tag{7.8}$$

$k = 0, 1, ..., m_N - 1$. The problem (7.4) defines a H-valued differential equation with random initial condition:

$$\frac{d}{ds}\widetilde{X}_{1t}^N(s) = S(t_{k+1}-s) f\left(s, \widetilde{X}_{1t}^N(s)\right), \quad s \in (t_k, t_{k+1}), \tag{7.9}$$

$$\widetilde{X}_{1t}^N(t_k) = X_{2t}^N(t_k - 0)$$

$$+ \int_0^{t_k} \left[S(t_{k+1}-u) f\left(u, \widetilde{X}_{1t}^N(u)\right) - S(t_k - u) f\left(u, \widetilde{X}_{1t}^N(u)\right)\right] du. \tag{7.10}$$

It follows from the properties of the semigroup $S(s)$ that

$$\|S(s)\| \leqq c \quad \forall s \in [0,T]. \tag{7.11}$$

Consequently,

$$\|S(t_{k+1}-s) f(s,x) - S(t_{k+1}-s) f(s,y)\|_H \leqq c_1 L \|x-y\|_H$$

and

$$\|S(t_{k+1}-s) f(s,x)\| \leqq c_1 L (1 + \|x\|_H)$$

for all $x, y \in H$, $s \in [t_k, t_{k+1})$. Therefore, (7.9) and (7.10) have a unique H-valued continuous solution $X_{1t}^N(s), s \in [t_k, t_{k+1})$. Since the initial condition is $\mathcal{F}_{t_k}$ -measurable, the solution process $X_{1t}^N(s), s \in [t_k, t_{k+1})$ is $\mathcal{F}_{t_k}$ -measurable. The process (7.4) is defined by $X_{1t}^N(t_{k+1} - 0)$ and Itô integrals. Consequently, we approximate the solution of the stochastic evolution equation (7.1) by the solutions of ordinary H-valued differential equations and the terms which can be modelled by H-valued Itô integrals. The convergence of the approximation holds in the following sense.

Theorem 6. $\lim_{N\to\infty} E\left\|X_{1t}^{N}(s) - X(s)\right\|_{H}^{2} = 0$ *for all* $s \in [0,t]$ *and* $t \in [0,T]$.

Proof. See Theorem 3.1 of Grecksch and Anh [15]. ∎

Often the temporal dependence displayed by (7.1) is not strong enough for certain applications. Examples include the transport and diffusion of molecules in porous media where the non-homogeneity of the media may result in the correlation function decaying much slower than the usual exponential rate. An alternative to (7.1) was proposed in Grecksch and Anh [14]. They proved the existence and uniqueness of a solution to the stochastic evolution equation

$$\begin{aligned} dX(t) &= A_x X(t)\,dt + f(t, X(t))\,dt + g(t)\,dB^h(t), \\ X(0) &= X_0, \quad X(t,x)\,|_{x\in\partial\Omega} = 0, \end{aligned} \tag{7.12}$$

where A_x is the infinitesimal generator of a strongly continuous semigroup and $B^h(t)$ is the Hilbert space-valued fractional Brownian motion defined by

$$B^h(t) = \sum_{k=1}^{\infty} B_i^h(t)\, e_i$$

with $B_i^h(t)$ being a sequence of independent fractional Brownian motions and $\{e_i\}$ is a complete orthonormal system in the separable Hilbert space. Here we may consider the special case $A_x = -\kappa(-\Delta)^{\alpha}(I-\Delta)^{\gamma}$. The solution to (7.12) is then given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s) f(s, X(s))\,ds + \int_0^t S(t-s) g(s)\,dB^h(s), \tag{7.13}$$

where $S(t)$ is the Riesz-Bessel semigroup. The splitting method described above can now be performed on (7.13) to obtain a time discretisation for (7.12). The mean-square convergence of this splitting method is given by Theorem 6 above.

8. Some extension

We have seen that the fractional diffusion equation (1.6) represents a true diffusion process with particles trajectories being given by a homogeneous Lévy motion. An extension of (1.6) to allow some form of temporal dependence is to introduce a fractional-in-time derivative so that the fractional diffusion equation becomes

$$\frac{\partial^{\beta} c}{\partial t^{\beta}} = -\kappa(-\Delta)^{\alpha}(I-\Delta)^{\gamma} c. \tag{8.1}$$

The Green function of (8.1) with fractional derivative in the Caputo-Djrbashian sense was identified in Anh and Leonenko [1] as

$$G(t,x) = \int_{\mathbb{R}^n} e^{i\langle\lambda,x\rangle} E_{\beta}\left(-\kappa t^{\beta}|\lambda|^{2\alpha}\left(1+|\lambda|^2\right)^{\gamma}\right) d\lambda, \tag{8.2}$$

where $E_{\beta}(x)$ is the one-parameter Mittag-Leffler function. This is an entire function with series expansion

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\beta k)}.$$

Using Theorem 1 of Feller [13, Chapter XIII-7], Theorem 1.3-6 of Djrbashian [10] and our Theorem 1 it follows that for $\alpha \in (0,1]$, $\beta \in (0,1]$ and $\alpha + \gamma \in [0,1]$

$$E_\beta \left(-\kappa t^\beta x^\alpha (1+x)^\gamma \right) \tag{8.3}$$

is the Laplace transform of an infinitely divisible distribution on $(0, \infty)$ and (8.2) is the density function of a type-G distribution.

Using the fact that the Green functions of (1.6) and (8.2) are densities of infinitely divisible distributions we could apply results from the time evolution properties of Lévy motion to study the qualitative behaviour of their solutions under some initial conditions. The simpliest application of this is the unimodal property. A finite measure μ is said to be strongly unimodal if $\mu * \rho$ is unimodal for any unimodal, finite measure ρ. A result of Watanabe [34] says that a Lévy motion is strongly unimodal for all times t if and only if it is Gaussian. Thus, if $c_0(x)$ is unimodal and $\int c_0(x)\,dx < \infty$, then $c(t,x)$ is unimodal for all t and all $c_0(x)$ only under the classical diffusion equation ($\alpha = 1, \beta = 1$ and $\gamma = 0$).

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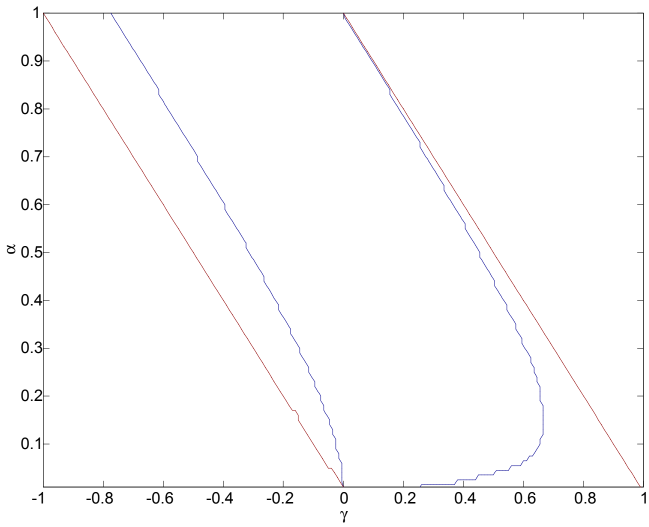

FIGURE 1. The regions for self-decomposability of the Riesz-Bessel-Lévy subordinator (see Remark 1).

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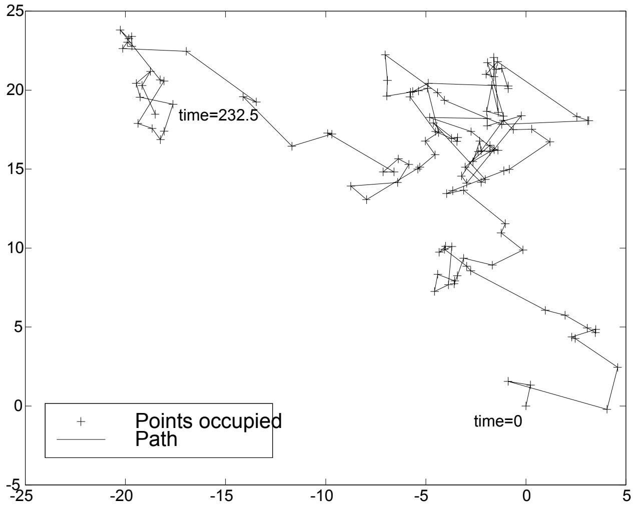

FIGURE 2. A simulated path of Riesz-Bessel-Lévy motion in $\mathbb{R}^2$ with $\alpha = 1$ and $\gamma = -1$.

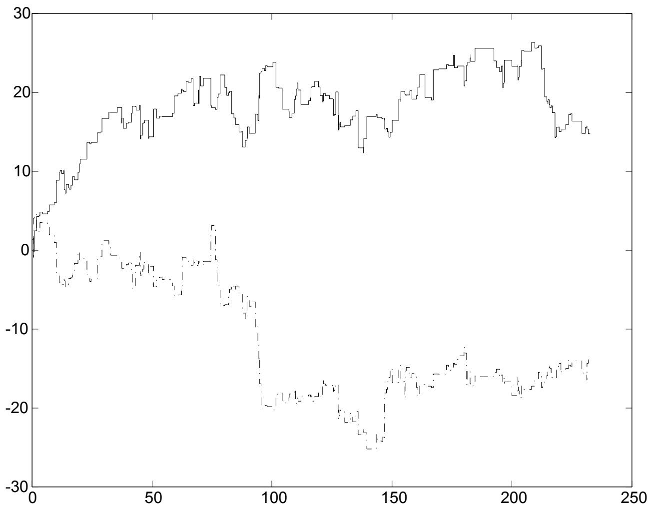

FIGURE 3. The x (dashed) and y (continuous) components over time of the path of Figure 1.

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