ON THE NUMBER OF ZEROS OF THE ABELIAN INTEGRALS
FOR A CLASS OF PERTURBED LIÉNARD SYSTEMS

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Addressing the weakened Hilbert’s 16th problem or the Hilbert-Arnold problem, this paper gives an upper bound $B(n) \leq 7n + 5$ for the number of zeros of the Abelian integrals for a class of Liénard systems. We proved the main result using the Picard-Fuchs equations and the algebraic structure of the integrals.

Mathematics Subject Classification: 34C05, 58F

1. Introduction

Research on limit cycle behaviour is important both for theoretical advances and practical applications [Arnold, 1990; Chow et al., 2002; Gavrilov, 2001; Giacomini et al., 1996; Han, 1997; Li, 2003; Smale, 1991; Roussarie, 1988]. The second part of Hilbert’s 16th problem relates to a difficult problem: the uniformly bounded upper bound of the number of limit cycles of planar polynomial differential systems. This is also called the purely existential problem by Dumortier et al., [1994].

To simplify the purely existential problem, Dumortier et al., also posed the so-called finite cyclicity method, known as the 121-program of Dumortier-Roussarie-Rousseau, for planar polynomial differential systems of order two [Dumortier et al., 1994]. Without the need to directly investigate the number of zeros of the Abelian integrals on the intervals of periodic solutions, the method focuses on the study of the finite cyclicity of limit period sets. This makes the problem easier to solve.
because periodic solutions have finite cyclicity. It has hoped that the existential problem for planar polynomial systems of order two is solved in this way. However, Ilyashenko [2002] has pointed out that the finite cyclicity method is unrealistic for general systems, even for cubic systems. This is because the method takes the advantage of the properties of quadratic vector fields but these properties cannot be easily generalized to higher-order vector fields. Therefore, solving the problem using the finite cyclicity method is not straightforward at all for general systems. Some weakened versions of the original problem, such as those in [Arnold, 1990; Smale, 1991], are still attractive to researchers.

This paper will address the weakened Hilbert’s 16th problem or the Hilbert-Arnold problem posed by Arnold, [1990]: Let \( H(x, y) \) be a real polynomial of order \( m + 1 \) and \( \Gamma_h \) be a continuous family of compact ovals defined by \( H(x, y) = h \), where \( h \) belongs to some open intervals. What is the maximal number \( B(m, n) \) of the zeros of the following integral?

\[
I(h) = \oint_{\Gamma_h} g(x, y)dx + f(x, y)dy,
\]

where \( f, g \) are polynomials in \( x, y \) and \( \max\{\deg f(x, y), \deg g(x, y)\} \leq n \). This integral is also called Abelian integral in [Li, 2003].

Investigations into the number of zeros of \( I(h) \) have received increasing interests, for instance, [Chow et al., 2002; Dumortier & Li, 2001, 2003; Gavrilov, 2001; Gavrilov 1998; Horozov & Ilive, 1994; Horozov & Ilive, 1998; Ilive 1998; Petrov, 1984-1990; Zhang, Chen & Zang, 2004; Zhang, Tadé & Tian, 2007]. Recently, Horozov & Ilive, [1998] gave an explicit upper bound \( B(2, n) \leq 5n + 15 \), and Chow et al., [2002] and Gavrilov, [2001] gave \( B(2, 2) \leq 2 \). Restricting to the Hamiltonian of the form \( H(x, y) = \frac{1}{2}y^2 + U(x) \) with \( \deg U(x) = m + 1 \), the number of zeros of the Abelian integrals \( I(h) \) has been estimated for \( m = 3 \) in a series of papers [Petrov, 1984-1990; Zhang, Tadé & Tian, 2007]. Dumortier & Li, [2001, 2003] have investigated the special case of \( m = 3 \) and \( n = 3 \).

Consider the following perturbed system

\[
\begin{align*}
\dot{x} &= y + \varepsilon f(x, y), \\
\dot{y} &= -(x^5 + bx^3 + x) + \varepsilon g(x, y),
\end{align*}
\]

where \( \varepsilon > 0 \) is small, and \( b \) is a negative constant satisfying \( b < -2 \). Obviously, when \( \varepsilon = 0 \), system (2) is equivalent to a Linéard system of the form

\[
\ddot{x} - \dot{x}Q(x) - P(x) = 0,
\]

whose Hamiltonian has the form

\[
H(x, y) = \frac{y^2 + x^2}{2} + bx^4 + x^6,
\]

where \( H = h_2 \) corresponds to the double eight loop, \( H = h, h \in (0, h_2) \) corresponds to the family of closed orbit surrounding the origin only, and \( H = h, h > h_2 \) corresponds to the family of closed orbit surrounding the double eight loop. When
Let $b = -5/2$, $f(x, y) = 0$, $g(x, y) = -(a_0 + a_1 x^2 + a_2 x^4)y$, H. Zang et al. [2004] obtained that the Hopf cyclicity is two and gave the new configurations of limit cycles bifurcated from the homoclinic or heteroclinic loops using the methods developed in [Han, 1997 ; Han & Chen, 2000 ; Han et al., 2004 ; Zhang, Zang & Han, 2004 ; Zhang, Han & Zang, 2004].

In this paper, we will study the Abelian integrals $I(h)$ of system (2) for the case of $b < -2$. Since $\oint_{\Gamma_h} x^{2k+1} y^l dx = 0$ for $h \in (0, h_2) \cup (h_2, +\infty)$, the Abelian integral becomes $I(h) = \int_{\Gamma_h} \sum_{k+l \leq n} a_{k,l} x^{2k} y^l dx$. Our main result is stated in the following theorem.

**Theorem 1.1.** Let $B(n)$ be the number of zeros of $I(h) = \int_{\Gamma_h} \sum_{k+l \leq n} a_{k,l} x^{2k} y^l dx$ in $(0, h_2) \cup (h_2, +\infty)$. Then we have

$$B(n) \leq 7n + 5.$$  

The rest of the paper will develop some fundamental results. Using these results, a formal proof will be given for this theorem.

### 2. The Picard-Fuchs equation and the algebraic structure of $I(h)$

Let $I_{k,l} = \oint_{\Gamma_h} x^{2k} y^l dx$, $I_0 = I_{0,1}$, $I_1 = I_{1,1}$, and $I_2 = I_{2,1}$. Then, a straightforward computating gives the following lemma.

**Lemma 2.1.** For $i = 0, 1, or 2$, $I_i$ satisfies the following Picard-Fuchs equation:

$$A \begin{pmatrix} I_0(h) \\ I_1(h) \\ I_2(h) \end{pmatrix} = (3hE + B) \begin{pmatrix} I'_0(h) \\ I'_1(h) \\ I'_2(h) \end{pmatrix},$$  

(4)

where $E$ is the identity matrix and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ \frac{b}{3} & 3 & 0 \\ \frac{(4-b^2)}{4} & \frac{3b}{4} & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & -\frac{b}{4} \\ \frac{b}{3} & 0 & \frac{(b^2-1)}{4} \\ 0 & \frac{(4-b^2)}{4} & \frac{(5b-b^2)}{4} \end{pmatrix}.$$  

Differentiating Eq. (4) yields

$$(A - 3E) \begin{pmatrix} I''_0(h) \\ I''_1(h) \\ I''_2(h) \end{pmatrix} = (3hE + B) \begin{pmatrix} I'''_0(h) \\ I'''_1(h) \\ I'''_2(h) \end{pmatrix}.$$  

(5)

**Lemma 2.2.** For $k + l = d \geq 3$, $I_{k,l}$ can be expressed as the linear combinations of $I_{i,j}$ (where $i + j = d - 1, d - 2$) and $hI_{i,j}$ (where $i + j = d - 2, i, 0, 1$).

**Proof.** Integrating by parts over $\Gamma_h$ and using $ydy + (x + bx^3 + x^5)dx = 0$, we have

$$I_{k,l} = \frac{2k - 5}{l+2} I_{k-3,l+2} - bI_{k-1,1} - I_{k-2,l}.$$  

(6)
By $H(x, y) = h$, we have

$$6I_{k+1,l} + 3bI_{k+1,l} + 2I_{k+1,l} = 12hI_{k-1,l}.$$  

(7)

Substituting Eq. (6) with $k \rightarrow k+2$ into Eq. (7), and then taking $k \rightarrow k+1, l \rightarrow l-2$ yield

$$\frac{6l + 4k + 2}{l} I_{k,t} + bI_{k+2,l-2} = 12hI_{k,l-2} - 4I_{k+1,l-2}.$$  

(8)

Taking $k = 0, 1, 2$ in Eq. (8) and $k = 3, 4, \cdots, d$ in Eq. (6), we obtain a linear algebraic system of the form

$$SJ = T, S = \left(\begin{array}{ccc} S_5 & 0 \\ 0 & E_{d-4} \end{array}\right),$$  

where $J = col(I_{0,d}, \cdots, I_{d,n}), E_{d-4}$ is an unit matrix of order $d - 4$ and

$$S_5 = \left(\begin{array}{cccc} \frac{6d+2}{d} & 0 & b & 0 \\ 0 & \frac{6d}{d-1} & 0 & b \\ 0 & 0 & \frac{6d-2}{d-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Since $\det S_5 \neq 0$ for $d \geq 3$ and $T$ contains only the integral terms $I_{i,j}$ (where $i + j = d - 1$ or $d - 2$) and $hI_{i,j}$ (where $i + j = d - 2, i = 0, 1, 2$), we complete the proof.

\[\square\]

**Lemma 2.3.** For $n \geq 3$, the integral

$$I(h) = \oint_{r_{d}} \sum_{k+l \leq n} a_{k,l} x^{2k} y^{l} dx, h \in (0, h_{2}) \cup (h_{2}, +\infty)$$

can be expressed as

$$I(h) = \alpha(h)I_{0} + \beta(h)I_{1} + \gamma(h)I_{2}$$  

(10)

where $\alpha, \beta, \gamma$ are polynomials in $h$ with $\deg \alpha = \left[\frac{n-1}{2}\right], \deg \beta = \left[\frac{n-2}{2}\right], \deg \gamma = \left[\frac{n-3}{2}\right]$, respectively. For $n = 0, \alpha = \beta = \gamma = 0$; for $n = 1, \alpha = 0, \beta = \gamma = 0$; for $n = 2, \deg \alpha = \deg \beta = 0, \gamma = 0$.

**Proof.** For $n = 0, 1, 2$ the assertion of lemma 2.3 is obvious.

Now, we will prove by the induction the case of $n \geq 3$. For $n = 3$, it follows from lemma 2.2 and the straightforward computation that

$$I(h) = (a_{0,1} + \frac{9}{2}a_{0,3}h)I_{0} + (a_{1,1} - \frac{3}{5}a_{03})I_{1} + (a_{2,1} - \frac{3}{20}a_{03}b)I_{2},$$  

(11)

which implies that the result of lemma 2.3 holds for $n = 3$.

Suppose for $n \leq d - 1, I^{n}(h) = \oint_{r_{d}} \sum_{k+l \leq n} a_{k,l} x^{2k} y^{l} dx$ can be expressed as

$$I^{n}(h) = \alpha^{n}(h)I_{0} + \beta^{n}(h)I_{1} + \gamma^{n}(h)I_{2}$$  

(12)
with \( \deg \alpha = \left\lfloor \frac{n-1}{2} \right\rfloor \), \( \deg \beta = \left\lfloor \frac{n-2}{2} \right\rfloor \), and \( \deg \gamma = \left\lfloor \frac{n-3}{2} \right\rfloor \).

For \( n = d \), by lemma 2.2 and the equality (12), we get

\[
I(h) = \oint_{\Gamma_n} \sum_{k+t \leq d} a_{k,t} x^{2k} y^t dx
= \sum_{k+t \leq d-1} a_{k,t} I_{k,t} + \sum_{k+t = d} a_{k,0} I_{k,0}
= \sum_{k+t \leq d-1} A_{k,t} I_{k,t} + \sum_{k+t = d-2} B_{k,0} I_{k,0}
= \alpha^{d-2} I_0 + \beta^{d-1} I_1 + \gamma^{d-1} I_2 + h[\alpha^{d-2} I_0 + \beta^{d-2} I_1 + \gamma^{d-2} I_2]
\equiv \alpha(h) I_0 + \beta(h) I_1 + \gamma(h) I_2,
\]

where

\[
\deg \alpha(h) \leq \max\{\deg \alpha^{d-1}, 1 + \deg \alpha^{d-2}\} \leq \max\{\left\lfloor \frac{d-2}{2} \right\rfloor, 1 + \left\lfloor \frac{d-3}{2} \right\rfloor\} = \left\lfloor \frac{d-1}{2} \right\rfloor,
\]

\[
\deg \beta(h) \leq \max\{\deg \beta^{d-1}, 1 + \deg \beta^{d-2}\} \leq \max\{\left\lfloor \frac{d-2}{2} \right\rfloor, 1 + \left\lfloor \frac{d-4}{2} \right\rfloor\} = \left\lfloor \frac{d-2}{2} \right\rfloor,
\]

\[
\deg \gamma(h) \leq \max\{\deg \gamma^{d-1}, 1 + \deg \gamma^{d-2}\} \leq \max\{\left\lfloor \frac{d-2}{2} \right\rfloor, 1 + \left\lfloor \frac{d-5}{2} \right\rfloor\} = \left\lfloor \frac{d-3}{2} \right\rfloor,
\]

which imply that \( \deg \alpha(h) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), \( \deg \beta(h) \leq \left\lfloor \frac{n-2}{2} \right\rfloor \), and \( \deg \gamma(h) \leq \left\lfloor \frac{n-3}{2} \right\rfloor \) hold for arbitrary \( n \). Similarly, we can prove by induction that there exist \( P_n^i = \sum_{k+t \leq n} a_{k,t} x^{2k} y^t \) (\( i = 0, 1, 2 \)) such that

\[
h^{\frac{n-1}{2}} I_0 = \oint_{\Gamma_n} P_n^0 dx,
\]

\[
h^{\frac{n-2}{2}} I_1 = \oint_{\Gamma_n} P_n^1 dx,
\]

\[
h^{\frac{n-3}{2}} I_2 = \oint_{\Gamma_n} P_n^2 dx.
\]

This completes the proof of lemma 2.3. \( \square \)

3. Linear estimate of the number of zeros of the Abelian integral

Let \( Z = \frac{3d}{4} I_1 + I_2 \). Then from Eqs. (4) - (5), we have

\[
\begin{pmatrix}
I_0 \\
I_1 \\
Z
\end{pmatrix}
= \begin{pmatrix}
d_{00} & d_{01} & d_{02} \\
d_{10} & d_{11} & d_{12} \\
d_{20} & d_{21} & d_{22}
\end{pmatrix}
\begin{pmatrix}
I'_0 \\
I'_1 \\
Z'
\end{pmatrix}
= D
\begin{pmatrix}
I'_0 \\
I'_1 \\
Z'
\end{pmatrix},
\]

(13)
Lemma 3.1.

where

\[ d_{00} = \frac{3}{2} h, \quad d_{01} = -\frac{1}{2} + \frac{3}{32} b^2, \quad d_{02} = -\frac{1}{4} b, \]
\[ d_{10} = -\frac{1}{8} b h, \quad d_{11} = h - \frac{9}{128} b^3 + \frac{3}{8} b, \quad d_{12} = \frac{3}{32} b^2 - \frac{1}{3}, \]
\[ d_{20} = \frac{3}{128} b^2 h - \frac{3}{8} h, \quad d_{21} = \frac{27}{2048} b^4 - \frac{9}{64} b^2 + \frac{3}{8}, \quad d_{22} = -\frac{9}{512} b^3 + \frac{5}{32} b + \frac{3}{4} h \]

and

\[ G(h) \begin{pmatrix} I_0^n \\ I_1^n \\ I_2^n \end{pmatrix} = \begin{pmatrix} f_{00} & f_{02} \\ f_{10} & f_{12} \\ f_{20} & f_{22} \end{pmatrix} \begin{pmatrix} I_0^n \\ I_1^n \\ I_2^n \end{pmatrix} \equiv F \begin{pmatrix} I_0^n \\ I_1^n \\ I_2^n \end{pmatrix}, \tag{15} \]

where \( G(h) = 12h(-144h^2 - 72bh + 3b^2 + 12b^3h - 16), \)

\[ f_{00} = 576 h^2 + (-36 b^3 + 192 b) h, \quad f_{02} = 12 b^2 - 48 h b h - 64, \]
\[ f_{10} = -144 b h^2 + (36 b^2 - 192) h, \quad f_{12} = 48 h (b^2 - 4), \]
\[ f_{20} = (36 b^2 - 576) h^2 + (27 b^3 - 144 b) h, \quad f_{22} = 12 h (-48 h + 3 b^3 - 16 b). \]

So, the integral \( I(h) \) reads

\[ I(h) = \alpha(h) I_0 + (\beta(h) - \frac{3b}{4} \gamma(h)) I_1 + \gamma(h) Z. \tag{17} \]

From Eqs. (13) and (17), we obtain

\[ I'(h) = \alpha_1(h) I_0' + \beta_1 I_1' + \gamma_1 Z', \tag{18} \]

where

\[ \alpha_1 = \alpha' d_{00} + (\beta' - \frac{3}{4} b \gamma') d_{10} + \gamma' d_{20} + \alpha, \]
\[ \beta_1 = \alpha' d_{01} + (\beta' - \frac{3}{4} b \gamma') d_{11} + \gamma' d_{21} + \beta - \frac{3}{4} b \gamma, \]
\[ \gamma_1 = \alpha' d_{02} + (\beta' - \frac{3}{4} b \gamma') d_{12} + \gamma' d_{22} + \gamma \]

and \( \deg \alpha_1 \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \deg \beta_1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor, \) \( \deg \gamma_1 \leq \left\lfloor \frac{n-3}{2} \right\rfloor. \)

Lemma 3.1. Let \( \Sigma = \{ h | h > 0, h \neq h_2 \}, \mathcal{N} = \{ h | \beta_1(h) = 0, h \in \Sigma \}. \) Then, we have the representation in \( \Sigma \setminus \mathcal{N} \)

\[ \begin{pmatrix} I'(h) \\ \beta_1(h) \end{pmatrix} = \frac{M(h)}{G(h) \beta_1^2(h)} \begin{pmatrix} I_0' \\ 1 \end{pmatrix}, \tag{19} \]

where

\[ M(h) = \alpha_2(h) I_0' + \beta_2(h) Z', \tag{20} \]

and \( \deg \alpha_2 \leq \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 2, \deg \beta_2 \leq \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 1, \alpha_2(0) = 0. \)
Let and lemma 3.1, it is easy to see that \( \text{deg} \beta_1 \) and \( \text{deg} \alpha_2 \leq \left[ \frac{n-1}{2} \right] + \left[ \frac{n-2}{2} \right] + 2 \). From Eq. (16), we have

\[ \text{deg} \alpha_2 \leq \left[ \frac{n-1}{2} \right] + \left[ \frac{n-2}{2} \right] + 1. \]

Since \( f_{00}(0) = f_{10}(0) = f_{20}(0) = G(0) = 0 \), we have \( \alpha_2(0) = 0 \).

**Lemma 3.2.** Let \( p, q, r \) be the number of zeros of \( I'(h), M(h) \), and \( \beta_1(h) \) in \((0, h_2)\), respectively. Then we have the following inequality:

\[ p \leq q + r + 1. \]

**Proof.** By Eqs. (15) and (18), we have in \( \Sigma \) \( \mathbb{N} \)

\[
\left( \frac{I'(h)}{\beta_1(h)} \right)' = \frac{1}{\beta_1^2} (\alpha_1 \beta_1 I''_0 + \beta_1^2 I''_1 + \gamma_1 \beta_1 Z'' + (\alpha'_1 \beta_1 - \beta'_1 \alpha_1) I'_0 + (\gamma'_1 \beta_1 - \beta'_1 \gamma_1) Z')
\]

\[
= \frac{1}{G(\beta_1)\beta_1} \left\{ (\alpha_1 \beta_1 f_{00} + \beta_1^2 f_{10} + \beta_1 \gamma_1 f_{20} + G(h)(\alpha'_1 \beta_1 - \beta'_1 \alpha_1)) I'_0 
+ (\alpha_1 \beta_1 f_{02} + \beta_1^2 f_{12} + \beta_1 \gamma_1 f_{22} + G(h)(\gamma'_1 \beta_1 - \beta'_1 \gamma_1)) Z' \right\}
\]

\[
= \frac{1}{G(\beta_1)\beta_1} (\alpha_2 I'_0 + \beta_2 Z').
\]

It is easy to see that \( \text{deg} \alpha_2 \leq \left[ \frac{n-1}{2} \right] + \left[ \frac{n-2}{2} \right] + 2, \text{deg} \beta_2 \leq \left[ \frac{n-1}{2} \right] + \left[ \frac{n-2}{2} \right] + 1 \). Since

\[ f_{00}(0) = f_{10}(0) = f_{20}(0) = G(0) = 0, \]

we have \( \alpha_2(0) = 0 \).

Denote by \( a_1, \cdots, a_r \) the element zeros of \( \beta_1 \) in the interval \([0, h_2]\) with \( 0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = h_2 \). Then around each \( a_j \), there exists a \( \delta \) such that there are no zeros of \( I'(h), M(h), \beta_1(h) \) in \([a_j - \delta, a_j + \delta]\) except \( a_j \).

In the interval \([a_j - \delta, a_j + \delta]\) with multiplicity \( l_j, m_j \), respectively. Suppose that \( a_j \) is the zero of \( \beta_1(h) \) with multiplicity \( k_j \). If \( k_j = l_j \), then we have \( m_j - 2l_j = 0 \), which implies \( m_j \geq l_j \). If \( k_j \neq l_j \), then we have \( l_j - k_j - 1 = m_j - 2k_j \), which also implies \( m_j \geq l_j \).

Denote by \( p_j \) and \( q_j \) the number of zeros of \( I'(h) \) and \( M(h) \) in \((a_j + \delta, a_{j+1} - \delta)\), respectively Then Eq. (20) implies that \( p_j \leq q_j + 1 \). Hence, we have the following estimate:

\[ p = p_0 + \sum_{j=1}^{r} (p_j + l_j) \leq q_0 + 1 + \sum_{j=1}^{r} (q_j + m_j + 1) = q + r + 1. \]

**Lemma 3.3.** Let \( W(h) = M(h)/I'_0 \). Then \( W(h) \) satisfies the Riccati equation:

\[
G(h)\beta_2 W(h) = R_2(h) + (G\beta'_2 + \beta_2 f_{22} - f_{00}) + 2\alpha_2 f_{02} W - f_{02} W^2,
\]

where \( R_2(h) = \beta_2(Ga'_2 + \beta_2 f_{20}) - \alpha_2(G\beta'_2 + \beta_2(f_{22} - f_{00})) - f_{02}\alpha'_2 + \text{deg} R_2 \leq 2\left[ \frac{n-1}{2} \right] + 2\left[ \frac{n-2}{2} \right] + 5, \) and \( R_2(0) = 0 \).

**Proof.** Let \( w = Z'/I'_0 \). Then from Eq. (15) we have

\[
Gw' = f_{20}(f_{22} - f_{00})w - f_{02} w^2
\]

and

\[
W = \frac{M(h)}{I'_0} = \alpha_2 + \beta_2 w.
\]

The straightforward computation and Eqs. (22)–(23) give Eq. (21). From Eq. (16) and lemma 3.1, it is easy to see that \( \text{deg} R_2 \leq 2\left[ \frac{n-1}{2} \right] + 2\left[ \frac{n-2}{2} \right] + 5, R_2(0) = 0 \).
Lemma 3.4. Assume \( R_2(h) \) and \( \beta_2(h) \) have \( s \) and \( t \) zeros in \((0, h_2)\), respectively. Then we have
\[
p \leq r + s + t + 2.
\]

**Proof.** Since \( I'_0 \neq 0 \), \( W(h) \) has just \( q \) zeros in \((0, h_2)\). We will split the proof into two steps.

1. Assume that \( \alpha_2 \) and \( \beta_2 \) have no common factor. The expression of \( M \) implies that the zeros of \( \beta_2 \) are not the zeros of \( M \). Let \( b_1, \cdots, b_t \) be all zeros of \( \beta_2 \) in \((0, h_2)\) with \( 0 = b_0 < b_1 < \cdots < b_t < b_{t+1} = h_2 \). Denotes by \( \sigma_j \) the number of zeros of \( R_2(h) \) in \((b_j, b_{j+1})\). Let \( \tilde{h}_1 \) and \( \tilde{h}_2 \) be the two consecutive zeros of \( W(h) \) in \((b_j, b_{j+1})\). Then \( W'(\tilde{h}_1)W'(\tilde{h}_2) \leq 0 \) so that \( R_2(\tilde{h}_1)R_2(\tilde{h}_2) \leq 0 \), which implies that between any two consecutive zeros of \( W(h) \) in \((b_j, b_{j+1})\) there is at least one zero of \( R_2(h) \). Hence, we have
\[
q \leq \sum_{j=0}^{t} (\sigma_j + 1) = s + t + 1.
\]
(24)

From lemma 3.2, we get
\[
p \leq r + s + t + 2.
\]

2. If \( \alpha_2 \) and \( \beta_2 \) have common factor \( N(h) \), then \( M(h) \) becomes
\[
M(h) = N(h)M_1(h), \quad M_1(h) = \alpha_3(h)I'_0 + \beta_3(h)Z'.
\]
For \( M_1(h) \), repeating the argument as above, we obtain the result. \( \square \)

Remark 3.5. By similar argument, we know that Lemma 3.2 and Lemma 3.4 are also true for \( h \in (h_2, +\infty) \).

4. Proof of Theorem 1.1

**Proof.**

1. For \( h \in (0, h_2) \). Since \( I(0) = R_2(0) \equiv 0 \), from Eq. (18) and lemmas 3.2-3.4 we get
\[
B_1(n) \leq r + s - 1 + t + 2
\]
\[
\leq \left[ \frac{n - 2}{2} \right] + 2\left[ \frac{n - 1}{2} \right] + 2\left[ \frac{n - 2}{2} \right] + 5 - 1 + \left( \left[ \frac{n - 1}{2} \right] + \left[ \frac{n - 2}{2} \right] + 1 \right) + 2
\]
\[
= 3\left[ \frac{n - 1}{2} \right] + 4\left[ \frac{n - 2}{2} \right] + 7 \leq \frac{7n + 3}{2}.
\]

2. For \( h > h_2 \). From Eq. (18) and lemmas 3.2-3.4, we get
\[
B_2(n) \leq r + s + t + 2 + 1
\]
\[
\leq \left[ \frac{n - 2}{2} \right] + 2\left[ \frac{n - 1}{2} \right] + 2\left[ \frac{n - 2}{2} \right] + 5 + \left( \left[ \frac{n - 1}{2} \right] + \left[ \frac{n - 2}{2} \right] + 1 \right) + 2 + 1
\]
\[
= 3\left[ \frac{n - 1}{2} \right] + 4\left[ \frac{n - 2}{2} \right] + 9 \leq \frac{7n + 7}{2}.
\]
On the Number of Zeros of the Abelian Integrals

The above two cases imply that the theorem holds.

In conclusion, an upper bound has been given for the number of zeros of the Abelian integrals for a class of perturbed linear systems. The result is proven using the Picard-Fuchs equations and the algebraic structure of the integrals.

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