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NUMERICAL SIMULATION OF THE NONLINEAR FRACTIONAL DYNAMICAL SYSTEMS WITH FRACTIONAL DAMPING FOR THE EXTENSIBLE AND INEXTENSIBLE PENDULUM *

C. YIN†, F. LIU‡ §, AND V. ANH‡

Abstract. In this paper we consider a class of nonlinear dynamical systems with fractional damping. The problem is transferred into a system of fractional-order differential equations. A computationally effective fractional predictor-corrector method is used to simulate and examine the effects and solution behavior of the nonlinear dynamical systems with fractional damping for extensible and inextensible pendulum. The corresponding error analysis is derived. Finally, some numerical examples are given. This method and technique can be applied to solve other fractional-order ordinary differential equations.

Key words. Nonlinear fractional differential equation, Predictor-Corrector method, error analysis, fractional derivative, the extensible and inextensible pendulum.

1. Introduction. Various fields of science and engineering deal with dynamical systems that can be described by fractional differential equations (FDEs) involving derivatives and integrals of non-integer order [9]. For example, we may cite system biology [16], physics [8], chemistry and biochemistry [17], hydrology application [6], fractional-order controllers [15], polymer rheology [12] and viscoelasticity [10]. Nonlinear dynamical systems with fractional damping also play an important role in engineering, seismic wave attenuation and polymer rheology [12]. Viscoelastic models involving fractional derivatives instead of ordinary derivatives are a new research issue. The most important aspect of the fractional derivatives is that it represent the singularity of a hereditary viscous kernel, and responsible for qualitative differences between singular and regular memory models. Several papers of fractional viscoelasticity are based on analytic methods, which are applicable to linear models [1, 10, 11].

Fractional differential equations have attracted the attention of many researchers. But analytic solutions of most FDEs are not usually given explicitly, in particular, for nonlinear fractional differential equations.

In this paper we consider the generic nonlinear initial-value problem with fractional damping:

$$au'' + \sum_{k=1}^{n} b_k D^{\alpha_k} u + cu' + \sum_{l=1}^{m} d_l D^{\beta_l} u + f(u) = g(t), \tag{1.1}$$

$$u(0) = u_0, u'(0) = u_1 \tag{1.2}$$

where $t \in [0, T]$, a, b_k, c, d_l, u_0, u_1 are constants, $0 < \beta_l < 1$, $1 < \alpha_k < 2$, and $0 < \beta_1 < \beta_2 < \cdots < \beta_m < 1$, $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 2$, $g(t)$ is a continuous

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†School of Mathematical Sciences, Xiamen University, Xiamen 361006, China (fwliu@xmu.edu.cn; yingeryin@126.com).

‡School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, QLD 4001, Australia (f.liu@qut.edu.au; v.anh@qut.edu.au).

§School of Mathematical Sciences, South China University of Technology, Guangzhou 510640, China.

function, $f(u)$ is Lipschitz-continuous with respect to u, $D^{\alpha_k}u$ and $D^{\beta_l}u$ denote the Caputo fractional derivatives of order α_k, β_l:

$$D_t^{\alpha}u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{u^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & 0 \le m-1 < \alpha < m, \\ \frac{d^m u(t)}{dt^m}, & m \in \aleph. \end{cases} \tag{1.3}$$

Wang [13] considered the (n, m) term fractional-order differential equation

$$D^{\frac{n}{m}}x(t) + a_1 D^{\frac{n-1}{m}}x(t) + \cdots + a_{n-1}D^{\frac{1}{m}}x(t) + a_n x(t) = u(t) \tag{1.4}$$

where $n, m \in N, n, m > 0, a_k, (k = 1, 2, \cdots, n)$ are arbitrary constants, $D^{\frac{k}{m}}$ is the Caputo fractional derivative.

The equation (4) is equivalent to the system of equations

$$\begin{cases} D^{\frac{1}{m}}x_1(t) & = & x_2(t), \\ D^{\frac{1}{m}}x_2(t) & = & x_3(t), \\ & \vdots & \\ D^{\frac{1}{m}}x_{n-1}(t) & = & x_n(t), \\ D^{\frac{1}{m}}x_n(t) & = & -a_1x_n(t) - \cdots - a_{n-1}x_2(t) - a_n x_1(t), \\ x(t) & = & x_1(t). \end{cases} \tag{1.5}$$

Diethelm [2] considered the general form of fractional differential equations

$$x^{(\alpha)}(t) = a_n D^{\beta_n}x(t) + a_{n-1}D^{\beta_{n-1}}x(t) + \cdots + a_1 D^{\beta_1}x(t) + f(t) \tag{1.6}$$

with $\alpha > \beta_n > \beta_{n-1} > \cdots > \beta_1$ and $\alpha - \beta_n < 1$, $\beta_j - \beta_{j-1} < 1$, $0 < \beta_1 < 1$.

Let M be the least common multiple of the denominators of $\beta_n, \beta_{n-1}, \cdots, \beta_1$ written in the form $\frac{n}{m}$, $n, m \in \aleph$ and let $\gamma = \frac{1}{M}$ and $N = M\alpha$. According to Diethelm and Ford [2], the equation (6) is equivalent to a system similar to the system (5). They discussed the existence and uniqueness of the solution, and proved the convergence and stability of the numerical methods based on a nearly equivalent system of fractional differential equations of order not exceeding β_n. Seredyńska [12] also used this technique for a generic nonlinear initial-value problem with fractional damping.

It is worth to point out that M is very large usually. It is well known that the system is difficult to solve when the number of a state variables becomes too large. To overcome this disadvantage, it is necessary to develop new techniques. We introduce a technique to transfer the multi-term fractional-order equation (1) into an equivalent system, which is the more general form (5). A numerical approximation is constructed by a decoupled method, then the predictor-corrector method is used for solving the fractional differential equation of a single order.

This paper is organized as follows. In Section 2, some basic ideas and lemmas are introduced. In Section 3, numerical simulation is proposed. In Section 4, error analysis is derived. Some numerical methods of simulating inextensible pendulum and extensible pendulum are presented in Sections 5 and 6, respectively. Finally, some numerical examples are given to evaluate the performance of the methods.

2. Basic ideas and lemmas. The basic analytical results on existence and uniqueness of solutions to fractional differential equations are given in [9, 12].

LEMMA 2.1. *Let* $\sum_{r=1}^{n} b_r t^{\alpha_r-1}/\Gamma(\alpha_r), \sum_{l=1}^{m} d_l t^{\beta_l-1}/\Gamma(\beta_l) \in L^1_{loc}$. *Equation (1) has a unique solution* $u \in W^1_{loc}$.

Proof. See Theorem 1 in [12]. □

LEMMA 2.2. *The differentiation operators* $D_t^{\alpha} f(t)$ *and* $D_t^m f(t)$ *satisfy the interchange rule:*

$$D_t^{\alpha}(D_t^m f(t)) = D_t^m(D_t^{\alpha} f(t)) = D_t^{\alpha+m} f(t),$$

$$f^{(s)}(0) = 0, s = n, n+1, \cdots, m; (m = 0, 1, \cdots; n-1 < \alpha < n).$$

Proof. See [9]. □

LEMMA 2.3. *Let* $f \in C^k[0,T]$ *for some* $T > 0$ *and some* $k \in \mathbb{N}$, *and let* $\beta \notin \mathbb{N}$ *such that* $0 < \beta < k$. *Then,* $D_t^{\beta} f(0) = 0$.

Proof. See [3]. □

Let $u \in C^2[0,T]$ for some $T > 0$. Using Lemma 2 and Lemma 3, we have

$$D_t^{\alpha-\beta}(D_t^{\beta} u) = D_t^{\alpha} u.$$

We can rewrite the generic nonlinear initial-value problem with fractional damping (1),(2) in the form of a system of fractional-order differential equations:

$$(2.1)\begin{cases} D_t^{\gamma_1} u_1(t) = D_t^{\beta_1} u_1(t) = u_2(t), \\ \vdots \\ D_t^{\gamma_m} u_m(t) = D_t^{\beta_m-\beta_{m-1}} u_m(t) = u_{m+1}(t), \\ D_t^{\gamma_{m+1}} u_{m+1}(t) = D_t^{1-\beta_m} u_{m+1}(t) = u_{m+2}(t), \\ D_t^{\gamma_{m+2}} u_{m+2}(t) = D_t^{\alpha_1-1} u_{m+2}(t) = u_{m+3}(t), \\ \vdots \\ D_t^{\gamma_{m+n+1}} u_{m+n+1}(t) = D_t^{\alpha_n-\alpha_{n-1}} u_{m+n+1}(t) = u_{m+n+2}(t), \\ D_t^{\gamma_{m+n+2}} u_{m+n+2}(t) = g(t) - f(u_1) - \sum_{l=1}^{m} d_l u_{l+1} - c u_{m+2} - \sum_{r=1}^{n} b_r u_{m+r+2}, \end{cases}$$

with initial conditions:

$$(2.2)\qquad u_1(0) = u_0^{(1)} = u_0, u_{m+2}(0) = u_0^{(m+2)} = u_1, u_i(0) = u_0^{(i)} = 0.$$

Where $2 \leq i \leq m+n+1, i \neq m+2$.

Using Lemmas 1, 2 and 3, we obtain the following theorem:

THEOREM 2.4. *The generic nonlinear initial-value problem with fractional damping (1),(2) is equivalent to the system of equations (7) together with the initial conditions (8).*

Proof. Using Lemma 1, we know the equation (1) has a unique solution.

Let $u_1(t) = u(t), u_2(t) = D_t^{\gamma_1} u_1(t) = D_t^{\beta_1} u_1(t) = D_t^{\beta_1} u(t)$,

$$u_3(t) = D_t^{\gamma_2} u_2(t) = D_t^{\beta_2-\beta_1} u_2(t) = D_t^{\beta_2-\beta_1}(D_t^{\beta_1} u_1(t)) = D_t^{\beta_2} u(t),$$

$$\vdots$$

$$u_{m+1}(t) = D_t^{\gamma_m} u_m(t) = D_t^{\beta_m - \beta_{m-1}} u_m(t) = D_t^{\beta_m} u(t),$$

$$u_{m+2}(t) = D_t^{\gamma_{m+1}} u_{m+1}(t) = D_t^{1-\beta_m} u_{m+1}(t) = u^{'}(t),$$

$$u_{m+3}(t) = D_t^{\gamma_{m+2}} u_{m+2}(t) = D_t^{\alpha_1 - 1} u_{m+2}(t) = D_t^{\alpha_1} u(t),$$

$$\vdots$$

$$u_{m+n+2}(t) = D_t^{\gamma_{m+n+1}} u_{m+n+1}(t) = D_t^{\alpha_n - \alpha_{n-1}} u_{m+n+1}(t) = D_t^{\alpha_n} u(t).$$

So equation (1) can be written as

$$\begin{aligned} D_t^{m+n+2} u_{m+n+2}(t) &= D_t^{2-\alpha_n} u_{m+n+2}(t) = u^{''}(t) \\ &= g(t) - f(u_1) - \sum_{l=1}^{m} d_l u_{l+1} - c u_{m+2} - \sum_{r=1}^{n} b_r u_{m+r+2}. \end{aligned}$$

It can be seen that $0 < \gamma_i < 1 (1 < i < m + n + 2)$.

Therefore, we obtain that the equation (1) is equivalent to the system of equations (7).

Let $u(0) = u_0, u_1(0) = u_0^{(1)}$. We have $u(0) = u_1(0)$. Thus

$$u_1(0) = u_0^{(1)}, u_{m+2}(0) = u_0^{m+2} = u^{'}(0) = u_1.$$

Using Lemma 3, we have

$$u_0^{(i)} = u_i(0) = D_t^{\gamma_{i-1}} u_{i-1}(0) = 0, (2 \le i \le m + n + 1, i \ne m + 2).$$

This completes the proof. □

3. Numerical simulation. In this section, a numerical technique for simulating the generic nonlinear initial-value problem with fractional damping (1) and initial conditions (2) is proposed.

Firstly, the generic nonlinear initial-value problem with fractional damping is decoupled, which is equivalent to solving the following equations:

$$\text{(3.1)} \qquad \begin{cases} D_t^{\gamma_1} u_1(t) = g_1(t, u_1), \\ D_t^{\gamma_2} u_2(t) = g_2(t, u_2), \\ \vdots \\ D_t^{\gamma_{m+n+2}} u_{m+n+2}(t) = g_{m+n+2}(t, u_{m+n+2}). \end{cases}$$

Secondly, we propose a computationally effective fractional predictor-corrector method for solving the following nonlinear initial-value problem:

$$\text{(3.2)} \qquad \begin{cases} D_t^{\gamma_i} u_i(t) = g_i(t, u_i), & (0 < \gamma_i < 1), \\ u_i(0) = u_0^{(i)}, & (i = 1, 2, \cdots, m + n + 2). \end{cases}$$

It is well known that the nonlinear initial-value problem (10) is equivalent to the Volterra integral equation:

$$\text{(3.3)} \qquad u_i(t) = u_0^{(i)} + \frac{1}{\Gamma(\gamma_i)} \int_0^t (t - \tau)^{\gamma_i - 1} [g_i(\tau, u_i(\tau))] d\tau.$$

We assume that we are working on a uniform grid $t_j = jh, j = 0, 1, \cdots M$, $T = Mh$, $u_i(t_j) = u_{i,j}$. The issue of stability is very important when implementing the method on a computer in finite-precision arithmetic because we must take into account the effects introduced by rounding errors. It is known that the classical Adams-Bashforth-Moulton method for first order ordinary differential equations is a reasonable and practically useful compromise in the sense that its stability properties allow for a safe application to mildly stiff equations without undue propagation of rounding error, whereas the implementation does not require extremely time consuming elements [5, 15]. Thus a fractional Adams-Bashforth method and a fractional Adams-Moulton method are chosen as our predictor and corrector formulae. The predictor is determined by the fractional Adams-Bashforth method [4, 15]:

$$u^P_{i,k+1} = u_0^{(i)} + \frac{1}{\Gamma(\gamma_i)} \sum_{j=0}^{k} b^{\gamma_i}_{j,k+1} u_{i+1,j}, \tag{3.4}$$

$$b^{\gamma_i}_{j,k+1} = \frac{h^{\gamma_i}}{\gamma_i}[(k+1-j)^{\gamma_i} - (k-j)^{\gamma_i}]. \tag{3.5}$$

The corrector formula is determined by the fraction Adams-Moulton method [4, 15]:

$$u_{i,k+1} = u_0^{(i)} + \frac{1}{\Gamma(\gamma_i)} \left(\sum_{j=0}^{k} a^{\gamma_i}_{j,k+1} u_{i+1,j} + a^{\gamma_i}_{k+1,k+1} u^P_{i+1,k+1} \right), \tag{3.6}$$

$$a^{\gamma_i}_{j,k+1} = \frac{h^{\gamma_i}}{\gamma_i(\gamma_i+1)} \begin{cases} k^{\gamma_i+1} - (k-\gamma_i)(k+1)^{\gamma_i}, & j = 0, \\ (k-j+2)^{\gamma_i+1} + (k-j)^{\gamma_i+1} & \\ \quad -2(k-j+1)^{\gamma_i+1}, & 1 \le j \le k, \\ 1, & j = k+1. \end{cases} \tag{3.7}$$

We then obtain the following fractional predictor-corrector method for solving the nonlinear initial-value problem (10).

Fractional predictor formulae:

$$u^P_{i,k+1} = u_0^{(i)} + \frac{1}{\Gamma(\gamma_i)} \sum_{j=0}^{k} b^{\gamma_i}_{j,k+1} u_{i+1,j}, (i = 1, 2, \cdots, m+n+1), \tag{3.8}$$

$$\begin{aligned} u^P_{m+n+2,k+1} &= u_0^{(m+n+2)} + \tfrac{1}{\Gamma(\gamma_{m+n+2})} \textstyle\sum_{j=0}^{k} b^{\gamma_{m+n+2}}_{j,k+1} \tfrac{1}{a}[g(t_j) \\ &- f(u_{1,j}) - \textstyle\sum_{l=1}^{m} d_l u_{l+1,j} - c u_{m+2,j} - \sum_{r=1}^{n} b_r u_{m+r+2,j}] \end{aligned} \tag{3.9}$$

Fractional corrector formulae:

$$u_{i,k+1} = u_0^{(i)} + \frac{1}{\Gamma(\gamma_i)} \left(\sum_{j=0}^{k} a^{\gamma_i}_{j,k+1} u_{i+1,j} + a^{\gamma_i}_{k+1,k+1} u^P_{i+1,k+1} \right), \tag{3.10}$$

where $i = 1, 2, \cdots, m+n+1$

$$
\begin{aligned}
u_{m+n+2,k+1} &= u_0^{(m+n+2)} + \tfrac{1}{\Gamma(\gamma_{m+n+2})} \sum_{j=0}^{k} a_{j,k+1}^{\gamma_{m+n+2}} \tfrac{1}{a}[g(t_j) - f(u_{1,j}) \\
&- \sum_{l=1}^{m} d_l u_{l+1,j} - c u_{m+2,j} - \sum_{r=1}^{n} b_r u_{m+r+2,j}] \\
&+ \tfrac{1}{\Gamma(\gamma_{m+n+2})} a_{k+1,k+1}^{\gamma_{m+n+2}} \tfrac{1}{a}[g(t_{k+1}) - f(u_{1,k+1}^P) \\
&- \sum_{l=1}^{m} d_l u_{l+1,k+1}^P - c u_{m+2,k+1}^P - \sum_{r=1}^{n} b_r u_{m+r+2,k+1}^P].
\end{aligned}
\tag{3.11}
$$

4. Error analysis. In this section we consider the error of our fractional predictor-corrector method.

LEMMA 4.1. *Let $u \in C^1[0,T]$. Then*

$$
\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\beta-1} u(t) dt - \sum_{j=0}^{k} b_{j,k+1}^{\beta} u(t_j) \right| \leq \frac{1}{\beta} \parallel u' \parallel_\infty t_{k+1}^{\beta} \cdot h. \tag{4.1}
$$

Proof. See [4, 14]. □

LEMMA 4.2. *If $u \in C^2[0,T]$, then there is a constant C_β depending only on β such that*

$$
\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\beta-1} u(t) dt - \sum_{j=0}^{k+1} a_{j,k+1}^{\beta} u(t_j) \right| \leq C_\beta \parallel u'' \parallel_\infty t_{k+1}^{\beta} \cdot h^2. \tag{4.2}
$$

Proof. See [4, 14]. □

THEOREM 4.3. *If $D_t^{\gamma_i} \in C^2[0,T]$, then*

$$
\max_{\substack{0 \leq j \leq M \\ 1 \leq i \leq n}} | u_i(t_j) - u_{i,j} | = O(h^q) \tag{4.3}
$$

where $q = 1 + \min_{1 \leq i \leq n} \gamma_i$

Proof. See [15]. □

From the above Theorem, the corresponding error can be estimated.

5. The inextensible pendulum with fractional damping. In this section, we propose a numerical simulation of the inextensible pendulum with fractional damping using the predictor-corrector method which has been described in Section 3. The inextensible pendulum with fractional damping is defined by the equation of motion (see [12]):

$$
\phi'' + (g/L)\sin\phi + \mu\tau^\alpha D^\alpha \phi + \lambda\tau\phi' + \nu\tau^\beta D^\beta \phi = 0, \tag{5.1}
$$

$$
\phi(0) = \phi_0; \quad \phi'(0) = \phi_1 \tag{5.2}
$$

with the elastic energy:

$$
E(t) = H(\phi(t), \phi'(t)) = \frac{1}{2} L^2 \phi'^2 + gL[1 - \cos(\phi)] \tag{5.3}
$$

where $0 \le t \le T, 0 < \beta < 1, 1 < \alpha < 2, D^{\alpha}\phi, D^{\beta}\phi$ are the Caputo fractional derivatives.

Let $\phi_1(t) = \phi(t)$ then we obtain

$$\left\{\begin{array}{l} D_t^{\alpha_1}\phi_1(t) = D_t^{\beta}\phi_1(t) = \phi_2(t), \\ D_t^{\alpha_2}\phi_2(t) = D_t^{1-\beta}\phi_2(t) = \phi_3(t), \\ D_t^{\alpha_3}\phi_3(t) = D_t^{\alpha-1}\phi_3(t) = \phi_4(t), \\ D_t^{\alpha_4}\phi_4(t) = D_t^{2-\alpha}\phi_4(t) = -(g/L)\sin\phi_1 - \mu\tau^{\alpha}\phi_4 - \lambda\tau\phi_3 - \nu\tau^{\beta}\phi_2, \end{array}\right. \tag{5.4}$$

with the following initial-value conditions:

$$\phi_1(0) = \phi_0^{(1)} = \phi_0, \phi_2(0) = \phi_0^{(2)} = 0, \phi_3(0) = \phi_0^{(3)} = \phi_1, \phi_4(0) = \phi_0^{(4)} = 0. \tag{5.5}$$

Using Theorem 1 we know that the inextensible pendulum with fractional damping (23) and (24) are equivalent to (26) and (27). The inextensible pendulum with fractional damping can be solved by using a decoupled technique and fractional predictor-corrector method.

Fractional predictor formulae:

$$\phi_{i,k+1}^P = \phi_0^{(i)} + \tfrac{1}{\Gamma(\alpha_i)} \sum_{j=0}^{k} b_{j,k+1}^{\alpha_i}\phi_{i+1,j}, \quad i = (1,2,3), \tag{5.6}$$

$$\phi_{4,k+1}^P = \phi_0^{(4)} + \frac{1}{\Gamma(\alpha_4)} \sum_{j=0}^{k} b_{j,k+1}^{\alpha_4}[-(g/L)\sin\phi_{1,j} - \mu\tau^{\alpha}\phi_{4,j} - \lambda\tau\phi_{3,j} - \nu\tau^{\beta}\phi_{2,j}]. \tag{5.7}$$

Fractional corrector formulae:

$$\phi_{i,k+1} = \phi_0^{(i)} + \frac{1}{\Gamma(\alpha_i)}\left(\sum_{j=0}^{k} a_{j,k+1}^{\alpha_i}\phi_{i+1,j} + a_{k+1,k+1}^{\alpha_i}\phi_{i+1,k+1}^P\right), (i = 1,2,3) \tag{5.8}$$

$$\begin{array}{rcl} \phi_{4,k+1} & = & \phi_0^{(4)} + \frac{1}{\Gamma(\alpha_4)} \sum_{j=0}^{k} a_{j,k+1}^{\alpha_4}[-(g/L)\sin\phi_{1,j} - \mu\tau^{\alpha}\phi_{4,j} \\ & - & \lambda\tau\phi_{3,j} - \nu\tau^{\beta}\phi_{2,j}] + \frac{1}{\Gamma(\alpha_4)} a_{k+1,k+1}^{\alpha_4}[-(g/L)\sin\phi_{1,k+1}^P \\ & - & \mu\tau^{\alpha}\phi_{4,k+1}^P - \lambda\tau\phi_{3,j}^P - \nu\tau^{\beta}\phi_{2,k+1}^P]. \end{array} \tag{5.9}$$

The elastic energy can be obtained by the following equation:

$$\begin{array}{rcl} E(t_{k+1}) & = & H(\phi(t_{k+1}), \phi^{'}(t_{k+1})) \\ & = & \frac{1}{2}L^2[\phi^{'}(t)]^2 + gL[1 - \cos(\phi(t_{k+1}))] \\ & = & \frac{1}{2}L^2[\phi_{3,k+1}]^2 + gL[1 - \cos(\phi_{1,k+1})]. \end{array} \tag{5.10}$$

6. The extensible pendulum with fractional damping. In this section, we propose numerical simulation for the extensible pendulum with fractional damping using predictor-corrector method of Section 3.

The extensible pendulum is a mass m suspended on a string in a gravitational field and allowed to move in a vertical plane. The spring is attached at the other end and is assumed to satisfy the linear Hook's law with the spring constant K. The rest length of the string is L and its actual length, expressed in terms of elongation ξ, equals the distance $r = L(1 + \xi)$ of the bob from the suspension point.

The equations of motion of a damped extensible elastic pendulum (see [12]) are:

$$\phi'' = -\left[\frac{g}{L}\sin(\phi) + 2\phi'\xi'\right]/(1+\xi) - (\mu\tau^{\alpha}D^{\alpha}\phi + \lambda\tau\phi' + \nu\tau^{\beta}D^{\beta}\phi)/(1+\xi)^2, \tag{6.1}$$

$$\xi'' = \frac{g}{L}\cos(\phi) + (1+\xi)\phi'^2 - \frac{K}{m}\xi - \mu_1\tau^{\alpha}D^{\alpha}\xi - \lambda_1\tau\xi' - \nu_1\tau^{\beta}D^{\beta}\xi, \tag{6.2}$$

where τ is a positive constant with the dimension $[T]$, while $\mu, \mu_1, \lambda, \lambda_1, \nu, \nu_1 \geq 0$ are dimensionless. The initial conditions are assumed as

$$\phi(0) = 1; \quad \phi'(0) = \xi(0) = \xi'(0) = 0. \tag{6.3}$$

The Hamiltonian is given by the formula:

$$\begin{aligned} H &= \tfrac{1}{2}[(1+\xi)^2\phi'^2 + \xi'^2] - (g/L)(1+\xi)\cos(\phi) + (K/mL^2)\xi^2/2 \\ &= \tfrac{1}{2}[p_\phi^2/(1+\xi)^2 + p_\xi^2] - (g/L)(1+\xi)\cos(\phi) + (K/mL^2)\xi^2/2, \end{aligned} \tag{6.4}$$

where $p_\xi := \xi', p_\phi := (1+\xi)^2\phi'$.

An extensible pendulum without damping is an interacting system consisting of two oscillators: the pendulum and the spring. The interaction is highly non-linear. The extension of the string affects the period of the pendulum, which is a parametric excitation. The angular deviation of the pendulum modulates the force acting on the string (see [12]).

Let $\phi_1(0) = \phi(0), \xi_1(0) = \xi(0)$, then we obtain

$$\begin{cases} {}_0^C D_t^{\alpha_1}\phi_1(t) = {}_0^C D_t^{\beta}\phi_1(t) = \phi_2(t), \\ {}_0^C D_t^{\alpha_2}\phi_2(t) = {}_0^C D_t^{1-\beta}\phi_2(t) = \phi_3(t), \\ {}_0^C D_t^{\alpha_3}\phi_3(t) = {}_0^C D_t^{\alpha-1}\phi_3(t) = \phi_4(t), \\ {}_0^C D_t^{\alpha_4}\phi_4(t) = {}_0^C D_t^{2-\alpha}\phi_4(t) = -\left[\frac{g}{L}\sin(\phi_1) + 2\phi_3\xi_3\right]/(1+\xi_1) \\ \qquad - (\mu\tau^{\alpha}\phi_4\lambda\tau\phi_3 - \nu\tau^{\beta}\phi_2)/(1+\xi_1)^2 \end{cases} \tag{6.5}$$

with the following initial-value conditions:

$$\phi_1(0) = \phi_0^{(1)} = 1, \phi_2(0) = \phi_0^{(2)} = 0, \phi_3(0) = \phi_0^{(3)} = 0, \phi_4(0) = \phi_0^{(4)} = 0, \tag{6.6}$$

and

$$\begin{cases} {}_0^C D_t^{\alpha_1}\xi_1(t) = {}_0^C D_t^{\beta}\xi_1(t) = \xi_2(t), \\ {}_0^C D_t^{\alpha_2}\xi_2(t) = {}_0^C D_t^{1-\beta}\xi_2(t) = \xi_3(t), \\ {}_0^C D_t^{\alpha_3}\xi_3(t) = {}_0^C D_t^{\alpha-1}\xi_3(t) = \xi_4(t), \\ {}_0^C D_t^{\alpha_4}\xi_4(t) = {}_0^C D_t^{2-\alpha}\xi_4(t) = \frac{g}{L}\cos(\phi_1) + (1+\xi_1)\phi_3^2 - \frac{K}{m}\xi_1 \\ \qquad - \mu_1\tau^{\alpha}\xi_4 - \lambda_1\tau\xi_3 - \nu_1\tau^{\beta}\xi_2 \end{cases} \tag{6.7}$$

with the following initial-value conditions:

$$\xi_1(0) = \xi_0^{(1)} = 0, \xi_2(0) = \xi_0^{(2)} = 0, \xi_3(0) = \xi_0^{(3)} = 0, \xi_4(0) = \xi_0^{(4)} = 0. \tag{6.8}$$

Fractional predictor formulae for $\phi_i(t)$:

$$\phi_{i,k+1}^P = \phi_0^{(i)} + \frac{1}{\Gamma(\alpha_i)}\sum_{j=0}^{k} b_{j,k+1}^{\alpha_i}\phi_{i+1,j}, \quad i = (1,2,3), \tag{6.9}$$

$$
\begin{aligned}
\phi^P_{4,k+1} &= \phi_0^{(4)} + \tfrac{1}{\Gamma(\alpha_4)} \sum_{j=0}^{k} b^{\alpha_4}_{j,k+1} \{ - \left[\tfrac{g}{L}\sin(\phi_{1,j}) + 2\phi_{3,j}\xi_{3,j} \right] / (1+\xi_{1,j}) \\
&- (\mu\tau^\alpha \phi_{4,j} + \lambda\tau\phi_{3,j} + \nu\tau^\beta \phi_{2,j})/(1+\xi_{1,j})^2 \}.
\end{aligned}
\tag{6.10}
$$

Fractional predictor formulae for $\xi_i(t)$:

$$
\xi^P_{i,k+1} = \xi_0^{(i)} + \tfrac{1}{\Gamma(\alpha_i)} \sum_{j=0}^{k} b^{\alpha_i}_{j,k+1} \xi_{i+1,j}, \quad i = (1,2,3),
\tag{6.11}
$$

$$
\begin{aligned}
\xi^P_{4,k+1} &= \xi_0^{(4)} + \tfrac{1}{\Gamma(\alpha_4)} \sum_{j=0}^{k} b^{\alpha_4}_{j,k+1} [\tfrac{g}{L}\cos(\phi_{1,j}) + (1+\xi_{1,j})\phi^2_{3,j} \\
&- \tfrac{K}{m}\xi_{1,j} - \mu_1\tau^\alpha \xi_{4,j} - \lambda_1\tau\xi_{3,j} - \nu_1\tau^\beta \xi_{2,j}].
\end{aligned}
\tag{6.12}
$$

Fractional corrector formulae for $\phi_i(t)$:

$$
\phi_{i,k+1} = \phi_0^{(i)} + \frac{1}{\Gamma(\alpha_i)} \left(\sum_{j=0}^{k} a^{\alpha_i}_{j,k+1}\phi_{i+1,j} + a^{\alpha_i}_{k+1,k+1}\phi^P_{i+1,k+1} \right), (i = 1,2,3),
\tag{6.13}
$$

$$
\begin{aligned}
\phi_{4,k+1} &= \phi_0^{(4)} + \tfrac{1}{\Gamma(\alpha_4)} \sum_{j=0}^{k} a^{\alpha_4}_{j,k+1} \{ - \left[\tfrac{g}{L}\sin(\phi_{1,j}) + 2\phi_{3,j}\xi_{3,j} \right] / (1+\xi_{1,j}) \\
&- (\mu\tau^\alpha \phi_{4,j} + \lambda\tau\phi_{3,j} + \nu\tau^\beta \phi_{2,j})/(1+\xi_{1,j})^2 \} \\
&+ \tfrac{1}{\Gamma(\alpha_4)} a^{\alpha_4}_{k+1,k+1} \{ - \left[\tfrac{g}{L}\sin(\phi^P_{1,k+1}) + 2\phi^P_{3,k+1}\xi^P_{3,k+1} \right] / (1+\xi^P_{1,k+1}) \\
&- (\mu\tau^\alpha \phi^P_{4,k+1} + \lambda\tau\phi^P_{3,k+1} + \nu\tau^\beta \phi^P_{2,k+1})/(1+\xi^P_{1,k+1})^2 \}.
\end{aligned}
\tag{6.14}
$$

Fractional corrector formulae for $\xi_i(t)$:

$$
\xi_{i,k+1} = \xi_0^{(i)} + \frac{1}{\Gamma(\alpha_i)} \left(\sum_{j=0}^{k} a^{\alpha_i}_{j,k+1}\xi_{i+1,j} + a^{\alpha_i}_{k+1,k+1}\xi^P_{i+1,k+1} \right), (i = 1,2,3),
\tag{6.15}
$$

$$
\begin{aligned}
\xi_{4,k+1} &= \xi_0^{(4)} + \tfrac{1}{\Gamma(\alpha_4)} \{ \sum_{j=0}^{k} a^{\alpha_4}_{j,k+1} [\tfrac{g}{L}\cos(\phi_{1,j}) + (1+\xi_{1,j})\phi^2_{3,j} \\
&- \tfrac{K}{m}\xi_{1,j} - \mu_1\tau^\alpha \xi_{4,j} - \lambda_1\tau\xi_{3,j} - \nu_1\tau^\beta \xi_{2,j}] \\
&+ a^{\alpha_4}_{k+1,k+1} [\tfrac{g}{L}\cos(\phi^P_{1,k+1}) + (1+\xi^P_{1,k+1})(\phi^P_{3,j})^2 \\
&- \tfrac{K}{m}\xi^P_{1,k+1} - \mu_1\tau^\alpha \xi^P_{4,k+1} - \lambda_1\tau\xi^P_{3,k+1} - \nu_1\tau^\beta \xi^P_{2,k+1}] \}.
\end{aligned}
\tag{6.16}
$$

The energy can be obtained by the following equation:

$$
\begin{aligned}
H &= \tfrac{1}{2}[(1+\xi_{1,k+1})^2\phi^2_{3,k+1} + \xi^2_{3,k+1}] \\
&- (g/L)(1+\xi_{1,k+1})\cos(\phi_{1,k+1}) + (K/mL^2)\xi^2_{1,k+1}/2 \\
&= \tfrac{1}{2}[p^2_\phi/(1+\xi_{1,k+1})^2 + p^2_\xi] \\
&- (g/L)(1+\xi_{1,k+1})\cos(\phi_{1,k+1}) + (K/mL^2)\xi^2_{1,k+1}/2
\end{aligned}
\tag{6.17}
$$

where $p_\xi := \xi_{3,k+1}, p_\phi := (1+\xi_{1,k+1})^2\phi_{3,k+1}$.

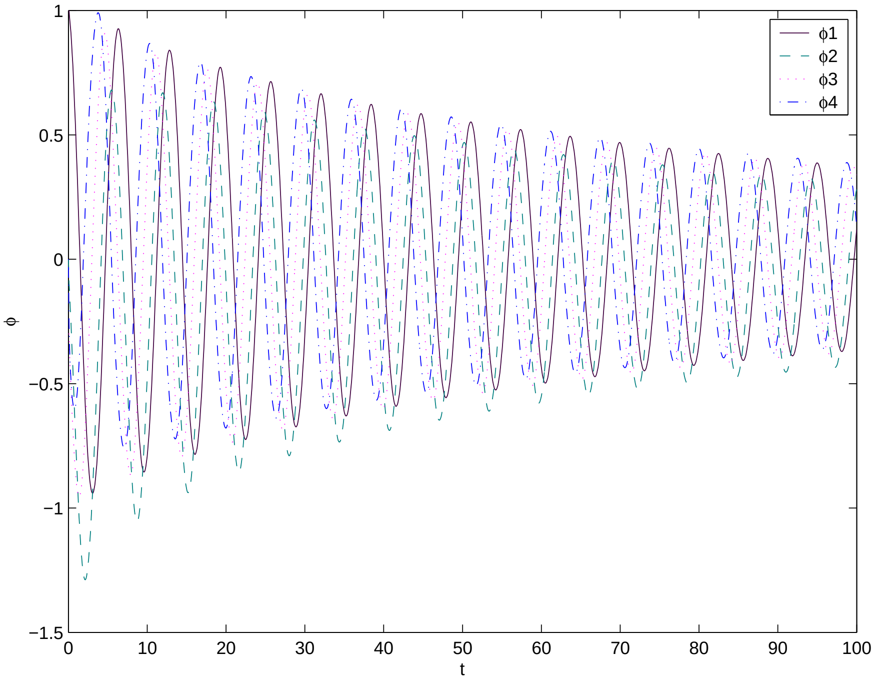

FIG. 7.1. *Inextensible pendulum, when* $\alpha = 1.5, \beta = 0.5$.

7. Numerical examples. In this section, two examples are given to demonstrate our theoretical analysis. The effects of fractional damping will be examined for the extensible and inextensible pendulum.

Example 1. Inextensible pendulum: We take $\phi(0) = 1.0$, $\phi^{'}(0) = -0.3$, $g/L = 1.0$, $\mu = 0.06$, $\tau = 0.8$, $\lambda = 0.1$, $\nu = 0.1$.

The simulating results in Example 1 are shown in Figures 1-4. Figure 1 shows the phase shift of oscillation of solution $\phi = \phi_1$ of the inextensible pendulum with $\alpha = 1.5$, $\beta = 0.5$ and the fractional derivatives ϕ_i of order 0.5, 1, 1.5. Figure 2 exhibits the elastic energy decay. From Figures 1-2, it can be seen that the elastic energy is not a monotonic decay. In this case it exhibits stationary points corresponding to the extremal positions of the pendulum. The fractional damping results in local energy minima at the extrema of the inextensible pendulum. These results are in good agreement with Seredyńska [12]. Figures 3 and 4 show the behaviors of the pendulum when $\alpha = 1.5$, β changes from 0 to 1 and $\beta = 0.5$, α changes from 1 to 2, respectively.

Example 2. Extensible pendulum: We take $\phi(0) = 1.0$, $\phi^{'}(0) = \xi(0) = \xi^{'}(0) = 0$, $g/L = 1.0$, $K/m = 1.3$, $\mu = 0.1$, $\tau = 0.8$, $\lambda = 0.1$, $\nu = 0.1$, $\mu_1 = 0.2$, $\lambda_1 = 0.1$, $\nu_1 = 0.1$.

An extensible pendulum is an interacting system consisting of two oscillators: the pendulum and the spring. The extension of the string affects the period of the pendulum, which is a parametric excitation. The simulating results in Example 2 are shown in Figures 5-10. Figure 5 shows the oscillations ϕ and ξ of the extensible pendulum with $\alpha = 1.5$, $\beta = 0.5$. Figure 6 shows the oscillation of solution $\phi = \phi_1$ of the extensible pendulum with $\alpha = 1.5$, $\beta = 0.5$ and the fractional derivatives ϕ_i of order 0.5, 1, 1.5. Figure 7 shows the oscillation of solution $\xi = \xi_1$ of the extensible

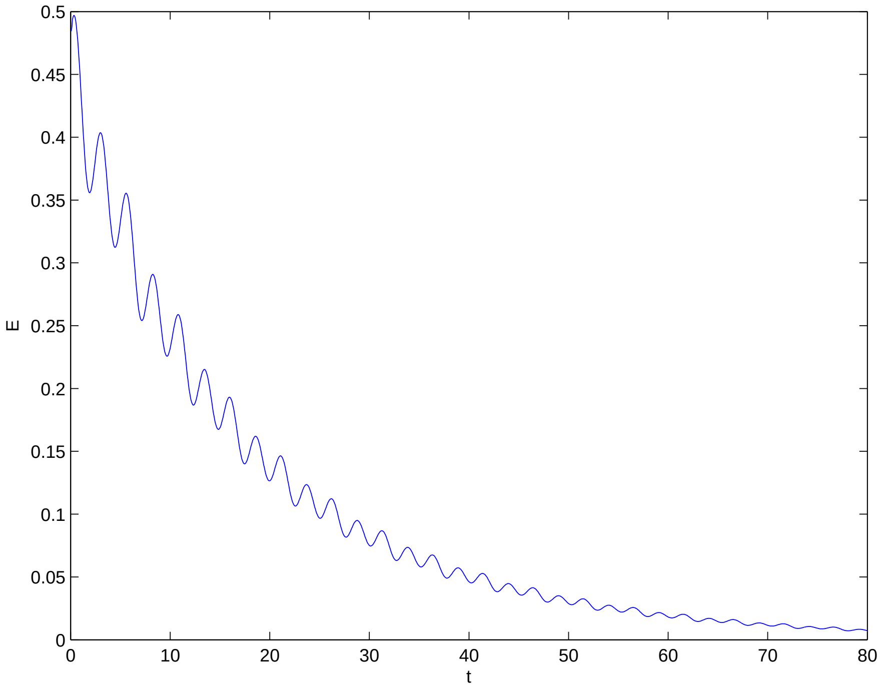

FIG. 7.2. *Energy of the inextensible pendulum, when* $\alpha = 1.5, \beta = 0.5$.

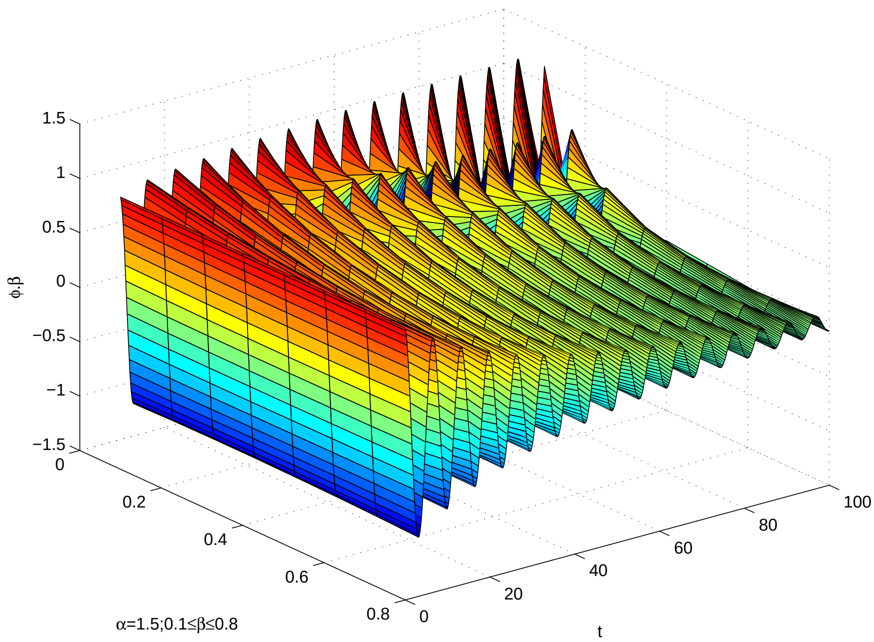

FIG. 7.3. *Inextensible pendulum, when* $\alpha = 1.5$, β *changes from* 0 *to* 1.

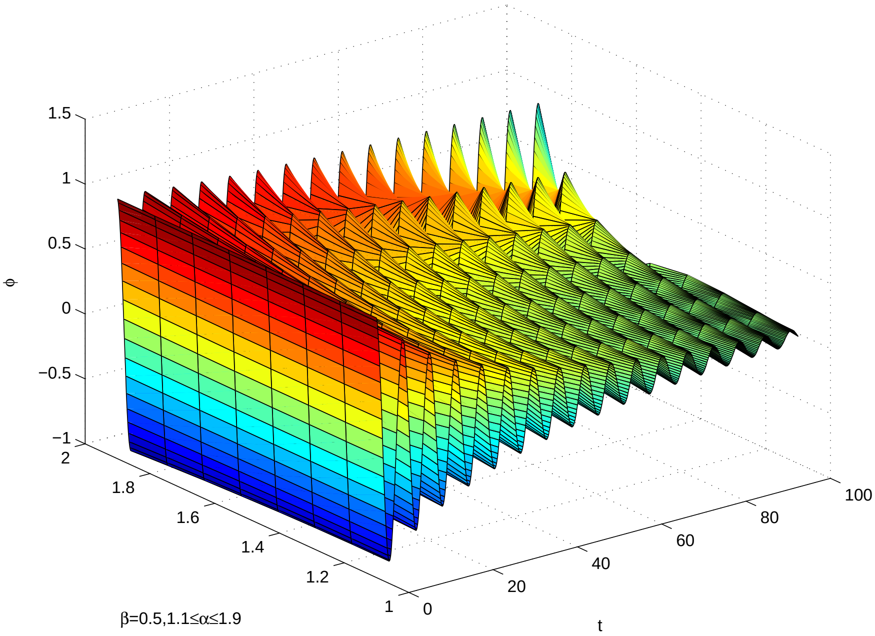

FIG. 7.4. *Inextensible pendulum, when $\beta = 0.5$, α changes from 1 to 2*

spring with $\alpha = 1.5$, $\beta = 0.5$ and the fractional derivatives ξ_i of order 0.5, 1, 1.5. From Figures 5-7, it can be seen that during the pendulum swing the gravitational force reaches its maximum four times. Resonant behavior is therefore expected if the ratio of the string characteristic frequency to the pendulum characteristic frequencies is 2:1. A resonant behavior is however observed in the extensible pendulum for arbitrary frequencies of the linearized systems. Energy decay for extensible pendulum with $\alpha = 1.5$, $\beta = 0.5$ is shown in Figure 8. From Figure 8, it can be seen that it is a non-monotone energy decay [12]. The behaviors of the extensible pendulum when β changes from 0 to 1 or α changes from 1 to 2 are shown in Figures 9 and 10, respectively. From Figures 9 and 10, we extract more property about the behaviors of the extensible pendulum with any fractional-order damping.

8. Conclusions. In this paper, a computationally effective fractional predictor-corrector method is used to simulate and examine the effects and solution behavior of the nonlinear dynamical systems with fractional damping for extensible and inextensible pendulum. Corresponding error analysis is derived. Some numerical examples are given to demonstrate that the numerical method is a computationally efficient method for this type of the systems. This method and technique can be applied to solve other fractional-order ordinary differential equations.

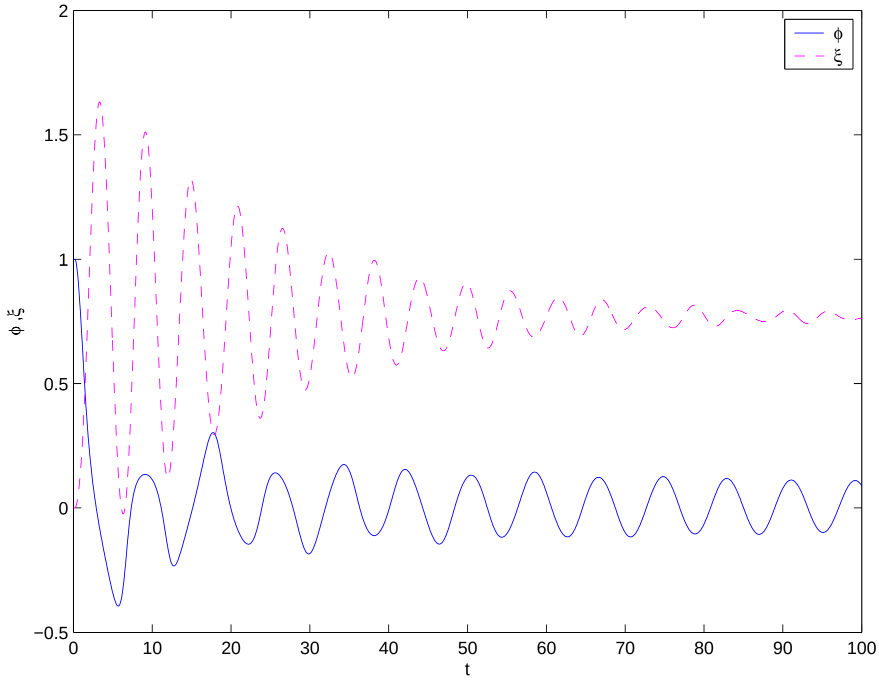

FIG. 7.5. *Extensible pendulum and it's elongation* ξ*, when* $\alpha = 1.5$*,* $\beta = 0.5$*.*

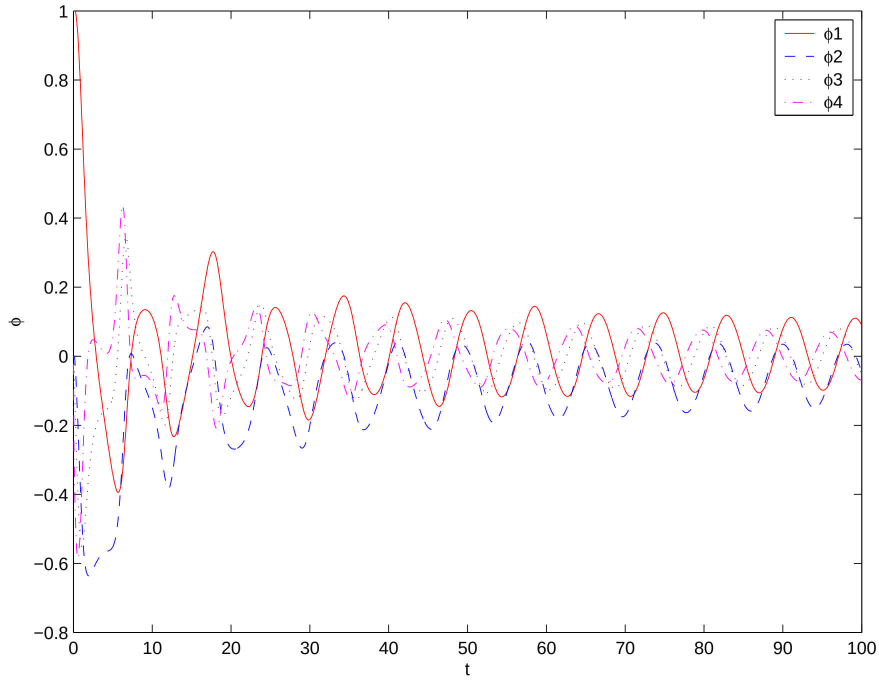

FIG. 7.6. *Extensible pendulum and the phase shift of the fractional derivatives* ϕ_i *of order 0.5, 1, 1.5., when* $\alpha = 1.5$*,* $\beta = 0.5$*.*

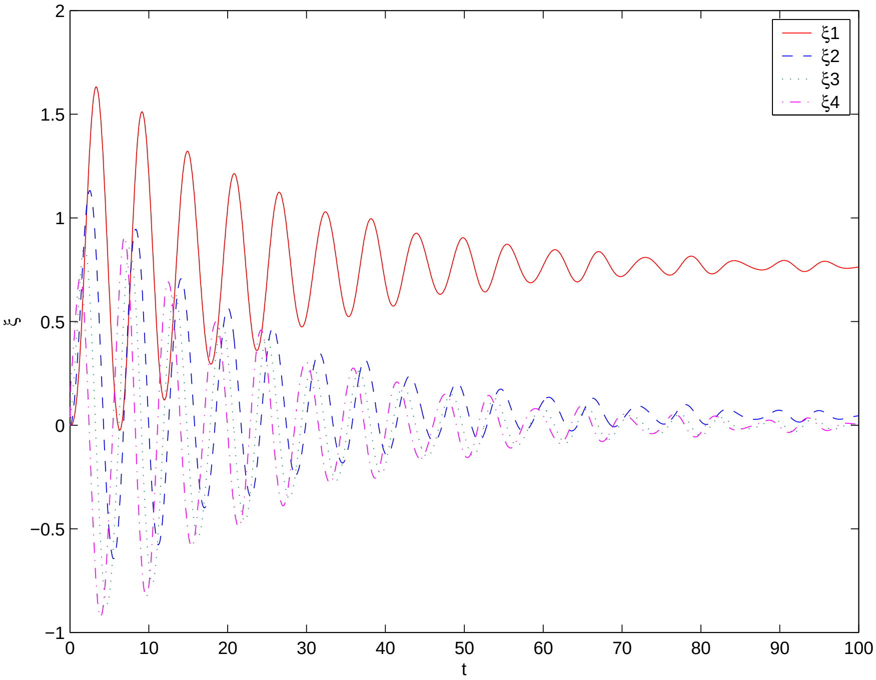

FIG. 7.7. *The elongation ξ of the extensible pendulum, when $\alpha = 1.5$, $\beta = 0.5$.*

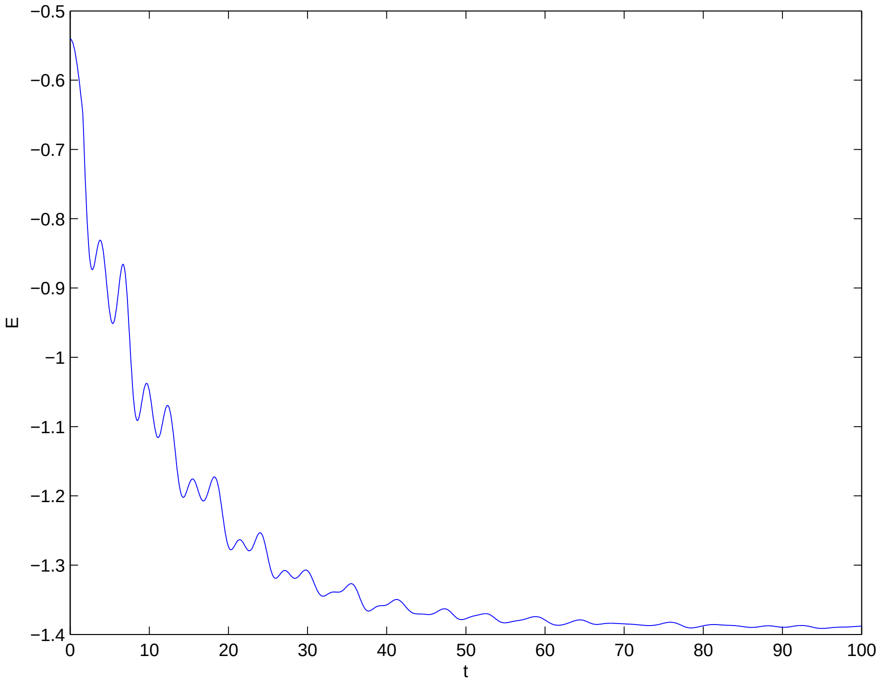

FIG. 7.8. *The energy of the extensible pendulum, when $\alpha = 1.5$, $\beta = 0.5$.*

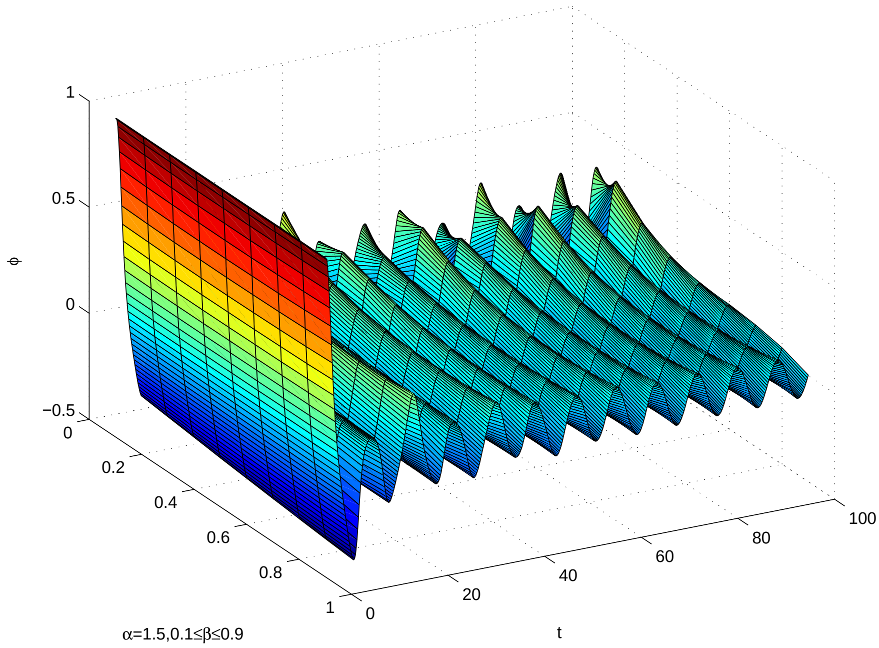

FIG. 7.9. *Extensible pendulum, when* $\alpha = 1.5$, β *changes from* 0 *to* 1.

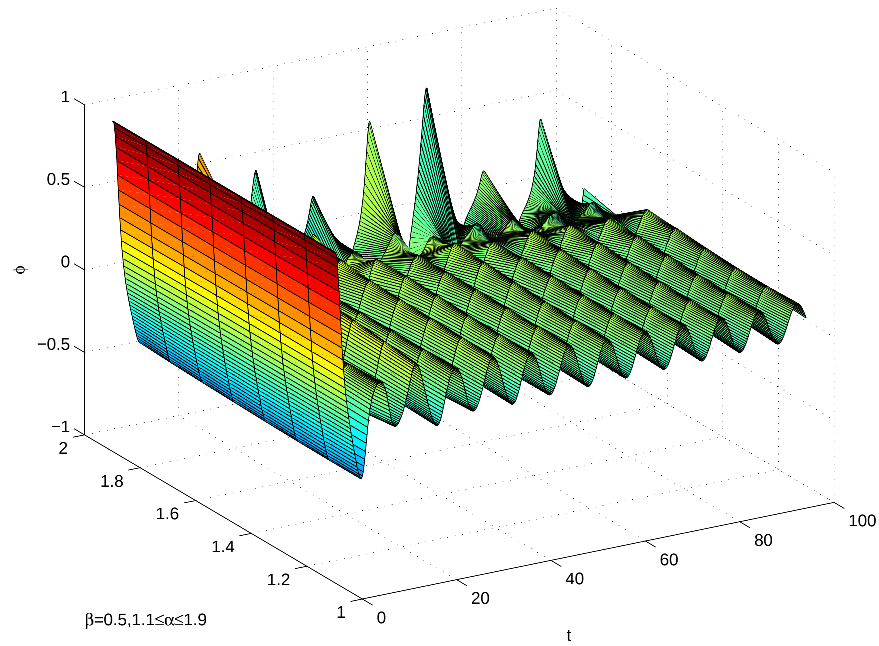

FIG. 7.10. *Extensible pendulum, when* $\beta = 0.5$, α *changes from* 1 *to* 2

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