PRE-SERVICE TEACHERS' MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE OF THE ORDER OF OPERATIONS

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Acronyms, BODMAS, connectionist, contextualised problems, convention, knowledge of content, knowledge of students' sense making, knowledge of teaching approaches, make sense, mathematical connections, mathematical expressions, mathematical pedagogical content knowledge, order of operations, pre-service teachers, properties of operations, stacked exponents, teaching approaches, transmission.

Abstract

Mathematical pedagogical content knowledge is an essential and complex facet of teacher knowledge that impacts on mathematics teaching and learning. It is defined as a form of teacher knowledge that bridges mathematical contents and the practice of teaching mathematics. This study focused on mathematical pedagogical content knowledge of pre-service secondary mathematics teachers and examined the ways in which they apply the order of operations and make mathematical connections across mathematical concepts, interpret students' written work in relation to the order of operations, and plan to teach the topic.

A case study research design methodology was used to generate and analyse data collected from 11 pre-service secondary mathematics teachers who were towards the end of their pre-service teacher education. Data generation methods included a questionnaire, task-based clinical interviews, and lesson plans in relation to the order of operations.

Findings provide further understanding of mathematical pedagogical content knowledge of pre-service secondary mathematics teachers. The participants' reasons for using the order of operations were mainly centred on the connections to relevant mathematical ideas, such as properties of operations, and the acronyms used to memorise the procedures. Misinterpretations in the order of operations that the participants encountered are discussed. Furthermore, analysis revealed three main approaches the participants used in interpreting students' written work: mathematical, pedagogical, and self-comparison. Difficulties that pre-service secondary mathematics teachers might experience when they lack the required content and pedagogical knowledge to analyse students' written work are highlighted. The results also revealed that the participants in this study adopted largely the transmission approach when planning to teach the order of operations. This study has shown that the conceptual framework used to represent mathematical pedagogical content knowledge provides theoretical and methodological value. The evidence suggests that this framework could be used for further research into mathematical pedagogical content knowledge beyond the teaching of the order of operations.

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List of Abbreviations

ACARA	Australian Curriculum, Assessment and Reporting Authority
BIDMAS	Brackets, Indices, Division, Multiplication, Addition, Subtraction
BODMAS	Brackets, Orders, Division, Multiplication, Addition, Subtraction
COVID-19	Coronavirus Disease 2019
DfE	Department for Education England
EPU	Malaysian Economic Planning Unit
KQ	Knowledge Quartet
MKT	Mathematical Knowledge for Teaching
MOE	Ministry of Education Malaysia
NCTM	National Council of Teachers of Mathematics
PEMDAS	Parentheses, Exponents, Multiplication, Division, Addition, Subtraction
PST	Pre-service secondary mathematics teacher
QUT	Queensland University of Technology
RME	Realistic Mathematics Education
TRU	Teaching for Robust Understanding
US	The United States

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Definitions

Connectionist

A connectionist approach emphasises dialogues between a teacher and students in which mathematical connections and negotiations of meanings are made explicit.

Contextualised problem

A contextualised problem refers to mathematical problems that reflect realistic situations, real or imaginable for individuals, and serve as a source of learning mathematics.

Discovery

A discovery approach emphasises learning more than teaching involving the teacher exploring options and alternatives put forward by students.

Knowledge of content

Knowledge of content involves the understanding of mathematical contents and procedures appropriate for teaching.

Knowledge of students' sense making

Knowledge of students' sense making refers to the knowledge used for recognising the ways students make sense of mathematical contents.

Knowledge of teaching approaches

Knowledge of teaching approaches concerns the knowledge about organising and representing mathematical contents in ways that students may understand.

Mathematical approach

Interpreting students' written work through a mathematical approach involves drawing on PSTs' existing knowledge and prior experience to interpret what students might be thinking.

Mathematical connection

A mathematical connection is a relationship between mathematical concepts.

Mathematical pedagogical content knowledge

Mathematical pedagogical content knowledge is a form of teacher knowledge that blends mathematical content and pedagogy required for teaching mathematics. In this study, mathematical pedagogical content knowledge consists of three knowledge components: knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches.

Order of operations

In mathematics, order of operations is a collection of rules used to simplify mathematical expressions.

Pedagogical approach

Interpreting students' written work through a pedagogical approach involves drawing on PSTs' pedagogical content knowledge. This includes knowing what students might commonly do and how a mathematical concept might generally be taught.

Self-comparison approach

Interpreting students' written work through a self-comparison approach involves PSTs reason how their own solutions are similar to or different from those of students.

Transmission

A transmission approach emphasises verbal teaching more than learning in which a clear introductory explanation is given followed by routine exercises.

Chapter 1: Introduction

The present study investigates mathematical pedagogical content knowledge in relation to the order of operations of pre-service secondary mathematics teachers (hereafter referred to as PSTs). Mathematical pedagogical content knowledge is a form of teacher knowledge that blends mathematical contents and the practice of teaching mathematics. This study is essential given that this specialised knowledge of teachers impacts upon teaching and learning of mathematics.

The first chapter of this thesis identifies the research problems and positions them within the context of the current focus of mathematical pedagogical content knowledge. This chapter includes discussions on research purposes, research questions to be explored, and the significance of the study. The chapter concludes with an outline of the remainder of this thesis.

1.1 THE PROBLEM

Despite the importance of teaching for understanding, researchers have found that there exists a discrepancy between what teachers intend to teach and what students understand (e.g., Lew et al., 2016; Thompson, 2013; Yoon, 2019). This discrepancy is attributable to the lack of congruence within pedagogical content knowledge, in particular, how much teachers know about the content, their students, and the ways to teach the content (Lo, 2020). This argument provokes the need to understand comprehensively the knowledge used in teaching if effective teaching and learning is to be achieved. In this regard, PST education is a key concern to ensure a well-developed pedagogical content knowledge (Hart et al., 2016).

In researching the literature on PST education related to mathematical pedagogical content knowledge, some global studies established teacher knowledge models, and some examined the knowledge associated with specific mathematical concepts. Shulman (1987) first introduced pedagogical content knowledge to describe the capacity of a teacher to transform content knowledge into forms that are pedagogically powerful yet adaptive to students. Since then, other researchers have drawn upon this teacher knowledge to describe knowledge needed for teaching mathematics (e.g., Askew et al., 1997; Ball et al., 2008; Baumert et al., 2010; Chick et al., 2006). Although these researchers have conceptualised mathematical pedagogical content knowledge differently, most of them include three main elements in their conceptualisations: understanding the specific content for teaching (knowledge of content), knowing how students learn (knowledge of students' sense making), and organising the content of teaching in a comprehensible manner to students (knowledge of teaching approaches). As different components of mathematical pedagogical content knowledge are not independent but intensively interact with each other (Hawkins, 2012), it is significant to examine all the knowledge components in the same research setting.

The global studies that generated empirical evidence to demonstrate PSTs' knowledge used in teaching mathematics have focused on specific mathematical topics or concepts. For example, such studies have conducted on fractions (Şahin et al., 2016), algebra (Baldinger, 2020; Tanisli & Kose, 2013), arithmetic and algebra word problems (van Dooren et al., 2002), decimals (Chick et al., 2006; Morris et al., 2009), and functions (Even & Tirosh, 1995). However, little is known about the knowledge required to approach the order of operations. In this regard, the order of operations is a set of rules used to determine which operation to perform before others (Papadopoulos, 2015). Hence, there is a clear need for more research into PSTs' mathematical pedagogical content knowledge in relation to the order of operations.

In understanding PSTs' knowledge of content, considerable research studies have made efforts to understand how PSTs make sense of a mathematical concept (e.g., Chin, 2013; Whitacre & Nickerson, 2016), but little attention has been paid to understand how mathematical procedures, like the order of operations, are made sense of. Although PSTs might consider the order of operations as arbitrary procedures, the conventions are indeed guided by reasons related to some other mathematical ideas, such as the properties of operations (Zazkis & Rouleau, 2018). Furthermore, there is a suggestion in the literature to understand how PSTs make sense of the order of operations of contextualised problems (Bay-Williams & Martinie, 2015; Cardone, 2015; Chang, 2019; Jeon, 2012). The present study is an attempt to contribute to these research gaps.

Mathematics teaching of the order of operations often deals with students' written work. Prior research in relation to analysing students' written work is largely restricted to describing students' procedures and checking students' answers for accuracy (Even & Tirosh, 1995; Gökkurt et al., 2013; Kiliç, 2011; Şahin et al., 2016; Shin, 2020; Tanisli & Kose, 2013; Tirosh, 2000). There is limited current research about the ways that PSTs used to analyse and interpret students' sense making through written work (Baldinger, 2020). Knowing how PSTs analyse students' written work is crucial because key features for supporting PSTs to develop abilities in interpreting students' sense making can be identified. The present study thus investigates how PSTs interpret students' sense making through their written work in relation to the order of operations.

A common way to teach the order of operations is using an acronym such as BODMAS (Brackets, Orders, Division, Multiplication, Addition, Subtraction) and its variants. However, PSTs misinterpreted the order of operations and made errors due to over reliance on the acronyms (Dupree, 2016; Glidden, 2008). As an alternative to acronyms, some pedagogical ideas were suggested such as using a hierarchical triangle (Ameis, 2011; Bay-Williams & Martinie, 2015), emphasising the connections between the order of operations and the properties of operations (Dupree, 2016; Jeon, 2012; Zazkis, 2018), and implementing problems in context (Bay-Williams & Martinie, 2015; Cardone, 2015; Holm 2021; Jeon, 2012). Given the possible approaches for teaching the order of operations, the present study aims to explore PSTs' teaching approaches in relation to the order of operations.

1.2 RESEARCH CONTEXT

This section narrows the focus from the global context and discusses the mathematical pedagogical content knowledge at a more local context in Malaysia. This is necessary because the professional knowledge of teaching mathematics may be slightly different due to differences in local institutions and cultural aspects. In 2013, the Ministry of Education in Malaysia introduced the National Education Blueprint containing a comprehensive transformation programme for the education system (Ministry of Education Malaysia [MOE], 2013). In this Education Blueprint, increasing the quality of pre-service teachers has been outlined as one of the aspirations in order to achieve the international education standards. In keeping pace with the international initial teacher education development, it is imperative to compare the standard of Malaysian pre-service teachers against international benchmarks.

The Teacher Education Development Study-Mathematics (TEDS-M), conducted by Tatto et al. (2008), was the first empirical international project of teacher preparation in 17 countries, including Malaysia. The TEDS-M provided substantial evidence for participating countries to compare the performance of PSTs and to explore opportunities to diversify strategies in improving teacher education. One of the findings of the TEDS-M showed that Malaysian PSTs performed below the international average for mathematics pedagogical content knowledge (Leong et al., 2015). In particular, PSTs were weak in analysing student errors due to limited content knowledge. The results culminated in a need to provide an in-depth analysis into the mathematics pedagogical content knowledge of Malaysian PSTs in order to identify both quality and poor aspects.

Given that the mathematics pedagogical content knowledge of the order of operations is unclear in the global context as well as in the local context in Malaysia, research into the knowledge required to prepare students in the learning of the order of operations becomes relevant. The current study attempts to contribute to this research gap in the existing literature.

The research site for the present study was a public university in Malaysia that focuses mainly on preparing future secondary teachers and has a long distinguished academic record in PST Education. The sample of this study was 11 PSTs who were towards the end of their PST education. The rationale for choosing final-year undergraduates as the target participants were that they had finished all the education professional courses of their Bachelor programs, and therefore were able to draw upon their knowledge gained over the period of the program.

The conceptual framework of this study is mathematical pedagogical content knowledge. It is conceptualised as a combination of three knowledge components, as outlined above: knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches. This framework has been specifically developed to shed light on the nature and extent of the mathematical pedagogical content knowledge utilised for teaching the order of operations. By employing this framework, this study can uncover both the strengths and limitations of PSTs' mathematical pedagogical content knowledge pertaining to the order of operations. Moreover, the use of this conceptual framework is anticipated to offer a unique perspective, not only in the analysis of the order of operations but also in initiating a broader discussion about various other mathematical concepts.

1.3 PURPOSES AND RESEARCH QUESTIONS

The overarching purpose of this study is to explore PSTs' mathematical pedagogical content knowledge of the order of operations. The mathematical pedagogical content knowledge emphasised in this study accounts for knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches. Accordingly, secondary purposes of this study, therefore, were to investigate reasons underlying PSTs' use of the order of operations, to identify PSTs' approaches in analysing and interpreting students' sense making of the order of operations, and to understand PSTs' ways to plan the teaching of the order of operations. Although mathematical pedagogical content knowledge developed in this study represents teacher knowledge for the order of operations, it would also apply to a range of other mathematical topics. To that end, there are five research questions to guide this study:

1. *How do pre-service secondary mathematics teachers apply the order of operations to evaluate mathematical expressions?*
2. *How do pre-service secondary mathematics teachers interpret the connections between the order of operations and the properties of operations?*
3. *How do pre-service secondary mathematics teachers determine the order of operations of contextualised problems?*
4. *How do pre-service secondary mathematics teachers interpret students' written work involving the order of operations?*
5. *How would pre-service secondary mathematics teachers plan to teach the order of operations?*

To answer these research questions, a case study research design was implemented. Data were gathered from a questionnaire, clinical task-based interviews, and lesson plans in relation to the order of operations. These data sources that were collected from 11 Malaysian PSTs provided rich qualitative data for analysis. Based on the developed framework of mathematical pedagogical content knowledge, the data were analysed thematically following the steps suggested by Ary et al. (2013) and using the approaches proposed by Braun and Clarke (2021).

1.4 SIGNIFICANCE OF THE STUDY

Existing studies on understanding PSTs' knowledge of the order of operations tend to focus on describing the order and checking the accuracy of answers (Dupree, 2016; Glidden, 2008; Pappanastos et al., 2002). The underlying reasons for computations have not been studied in as much detail. Specifically, the ways PSTs used to make sense of the order of operations have rarely been studied. This study will thus make a significant step forwards accounting for PSTs' sense making for the seemingly arbitrary order convention.

Past research has documented mathematical connections in other mathematical topics (Eli et al., 2013; Gamboa et al., 2020; García-García & Dolores-Flores, 2021; Hatisaru, 2022), but none in the order of operations. Making mathematical connections to the order of operations is important for PSTs so that they can help their students to make sense of the reasons for the convention. The present study will therefore extend the work of mathematical connections to understand how PSTs link the procedural rules within mathematics.

Analysing students' written work is an essential practice of mathematics teaching (Ergene & Bostan, 2022). Much can be gained from helping PSTs analyse students' written work, but such work should be based on an understanding of the process through which PSTs analyse students' written work and interpret students' sense making (Baldinger, 2020). Hence, the present study will provide empirical evidence on the approaches PSTs use to interpret students' sense making of the order of operations.

Evidence to show pedagogical aspects of the order of operations is relatively limited (Ameis, 2011; Zazkis, 2018). Although researchers outlined several pedagogical ideas to replace the acronyms that might cause misinterpretations in the order of operations, many of which were not supported by empirical evidence. This study will thus contribute to the body of empirical evidence about the teaching approaches of the order of operations planned by PSTs.

1.5 CONDUCTING THE STUDY AMIDST THE COVID-19 PANDEMIC

The global transmission of Coronavirus Disease 2019 (COVID-19) has led to devastating consequences such as loss of life, international border closures, economic shutdowns, school closures, and unprecedented challenges to human lives (Chin et al., 2022a). This study was conducted amidst of the COVID-19 pandemic, which initially posed obstacles to data collection due to school closures. However, in response to these circumstances, the method of data collection was adapted, transitioning from on-site data collection to a virtual method using the Zoom platform. To complement the change of method and to guarantee the contribution of the study, the literature review in Chapter 2 was conducted in a more comprehensive manner covering relevant studies conducted in Malaysia.

In addition, the methodology in Chapter 3 underwent several iterations and refinements, incorporating feedback from ethics advisors. Although teaching observations were not possible during the school closure, the study utilised pre-existing PSTs' lesson plans as an additional data collection tool. In conclusion, the COVID-19 pandemic did not hinder the novel contribution of this study, as diligent measures were taken to adapt and mitigate the effects of the pandemic on the research process.

1.6 THESIS OUTLINE

This thesis is organised into seven chapters. Chapter 1 introduces the research topic and defines the context of the study based on the research problems. The study's purposes, research questions, and significance are presented. Chapter 2 provides an analysis of the literature pertaining to mathematical pedagogical content knowledge of the order of operations. Chapter 3 establishes the conceptual framework to guide the study. Chapter 4 establishes the methodological theory and processes used in this study and provides a rationale for implementing a case study research design. Data collection methods, data analysis, issues of trustworthiness, and ethical considerations are outlined. The pilot study is also presented in this chapter. Chapters 5 provides documentation of the results of the questionnaire, interviews, and lesson plans. Chapter 6 discussed the results pertaining to each of the research questions outlined in Section 1.3 (p. 5). Chapter 7 contains a summary of the study and contributions to the field of mathematics education. The chapter ends with the limitations of the study and directions for future research.

Chapter 2: Literature Review

Chapter 2 reviews pertinent literature about PSTs' mathematical pedagogical content knowledge, with a particular focus on the order of operations. It begins by presenting a brief historical background and importance of the order of operations. The chapter then introduces the order of operations as a knowledge of content. This includes recognising difficulties of the order of operations. Mathematical connections in relation to the order of operations and approaches for interpreting students' written work are also discussed. Another discussion on different pedagogical ideas and teaching approaches of the order of operations is presented. This chapter also reviews past research studies conducted on PSTs' knowledge in teaching mathematics. Each of these sections has been included for its relevance to examine mathematical pedagogical content knowledge and the subsequent identification of research gaps to be filled in this study.

2.1 HISTORICAL BACKGROUND AND IMPORTANCE OF THE ORDER OF OPERATIONS

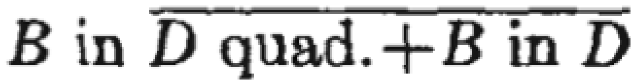
In mathematics, order of operations defines the sequence in which a mathematical expression involving multiple operations can be simplified (Zazkis & Marmur, 2018). This section presents a brief historical background of the order of operations. It also discusses the importance of the order of operations to mathematics education and to our daily life.

2.1.1 Historical Background

An analysis of the history of the order of operations reveals that not much is described in written documents about the origin of this mathematical idea. There is no exact information on when, where, and who invented the order of operations (Cajori, 1993; Peterson, 2019). However, there are traces of using the order of operations back in 17th century. This can be seen in Van Schooten's 1646 edition of Vieta, as presented in **Figure *2.1***.

Figure 2.1

Van Schooten's 1646 Edition of Vieta That Represents $B(D^2 + BD)$



$B \text{ in } \overline{D \text{ quad.} + B \text{ in } D}$

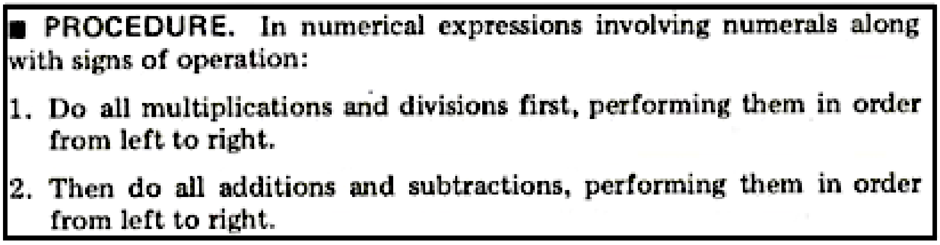
At that time, the use of a vinculum represented parentheses and the word, in, indicated the operation of multiplication. As shown in **Figure *2.1***, the vinculum grouped the two terms so that they are added before multiply by B. Nowadays, we represent this Van Schooten's notation as:

$$\mathrm{B(D^2 + BD)}$$

Only until the late 1800s, the term, Order of Operations, was starting to get used in textbooks (Vanderbeek, 2007). An example is the mathematics textbook, AMSCO's Integrated Mathematics Course 1 that included the order of operations in Chapter 4 (Dressler & Keenan, 1980, as cited in Joseph, 2014). It clearly stated the procedures to simplify mathematical expressions which involve four basic arithmetic operations (see **Figure *2.2***).

Figure 2.2

AMSCO's Integrated Mathematics Course 1 Textbook



■ **PROCEDURE.** In numerical expressions involving numerals along with signs of operation:

1. Do all multiplications and divisions first, performing them in order from left to right.
2. Then do all additions and subtractions, performing them in order from left to right.

In fact, in the 1920s, mathematicians were debating an issue regarding whether multiplication should take precedence over division (Cajori, 1993). However, no evidence was found on how they resolved the issue.

Without knowing much about the history, Peterson (2019) argued that the order of operations could have existed before algebraic notation existed as the order of operations lends itself well to writing polynomials with as few parentheses as possible. Imagine an algebraic expression looks like this,

$$\Big(\big((3 \times (x^2)) - (4 \times x)\big) + 7\Big)$$

This expression can be challenging to understand due to the presence of multiple pairs of parentheses. By applying the order of operations, the expression can be written in a simplified form as follow:

$$3 \times x^2 - 4 \times x + 7$$

or

$$3x^2 - 4x + 7$$

2.1.2 The Importance of the Order of Operations

Essentially, it is important that students learn to follow the order of operations so that they can evaluate expressions or solve equations based on this convention (Knill, 2014; Villiers, 2015). In addition to evaluate expressions, the rules of order of operations are used to avoid ambiguity, to ensure formation of succinct expressions and to maintain consistent communications among everyone. Take $2 + 3 \times 4$ as an example. If operations are performed from left to right, performing addition before multiplication:

$$\begin{aligned} 2 + 3 \times 4 &= 5 \times 4 \\ &= 20 \end{aligned}$$

the final answer is 20. If multiplication is operated first, then addition:

$$\begin{aligned} 2 + 3 \times 4 &= 2 + 12 \\ &= 14 \end{aligned}$$

the final answer is 14. Both order of computations resulted in different answers. Thus, it is essential to have a set of rules to come to a consistent outcome.

Scientific calculators are programmed to follow the order of operations (Joseph, 2014). As a school teacher, I experienced students who argued that the knowledge of order of operations is of no use since the scientific calculators are programmed to give a correct answer. This, however, is not completely true. The way some expressions are written is not the same as the way they are entered into a scientific calculator. As an illustration, the symbol $\wedge$ is entered into a scientific calculator to indicate exponentiate. When the expression 2^{3-1} is entered as $2 \wedge 3 - 1$ into a scientific calculator, it would give an erroneous answer, which is seven. An explanation for this error is that the scientific calculator first exponentiates then subtracts. In the expression 2^{3-1}, the subtraction is within the exponent so it should be entered into the scientific calculator as $2 \wedge (3 - 1)$ so that subtraction can be perform first. This signifies the necessity of the knowledge of order of operations, in particular knowledge about parentheses.

The order of operations is fundamental to advanced topics, such as algebra and composite functions (Banerjee, 2011). Take a composite function for an example. $f \circ g(x) = f(g(x))$ is read as f composed with g at x is equal to f of g of x. In this respect, combining functions in which the output of a function is the input of another function yields a composite function. To evaluate this composite function, it is crucial to follow the order of operations in which we must start with the innermost parentheses then follow by the outer parentheses.

The knowledge of order of operations has extensive use in everyday life (Joseph, 2014). Most of the everyday problems require the use of more than one operation. Take for instance, Andy purchased three large Hawaiian Chicken Pizzas that cost $\$12.50$ each, and he wants to split the cost among five people equally. He enters $12.50 + 12.50 + 12.50 \div 5$ into a scientific calculator and gets 27.50. Based on this calculation, each person needs to pay $\$27.50$ which does not make sense. In this case, the scientific calculator has divided only the last 12.50 by five. Indeed, Andy would have to include parentheses to calculate the total cost first: $(12.50 + 12.50 + 12.50) \div 5$, then he will get 7.50 which means each person needs to pay $\$7.50$. This illustrates the importance of using the correct order of calculations in order to get the correct answer.

2.2 THE ORDER OF OPERATIONS

Order of operations is a collection of rules used to simplify expressions in mathematics (Papadopoulos, 2015). The rules are expressed as:

i. First, perform operations in parentheses,

ii. Next, evaluate expressions with exponents,

iii. Next, execute multiplication and division from left to right,

iv. Last, execute addition and subtraction from left to right.

Typically, the order of operations is introduced in the middle grades (e.g., Australian Curriculum, Assessment and Reporting Authority [ACARA], 2018; Ministry of Education Malaysia [MOE], 2016; National Council of Teachers of Mathematics [NCTM], 2006). In the Malaysian mathematics curriculum, for example, the order of operations is formally introduced in Grade 7. The learning standards related to the order of operations are threefold (see **Table 2.1**). First , the standards focus mainly on following steps to evaluate mathematical expressions. Second, the properties of operations are to be described without linking how they are connected to the order of operations, and third, situations are used only as context for further practice of calculations.

Table 2.1

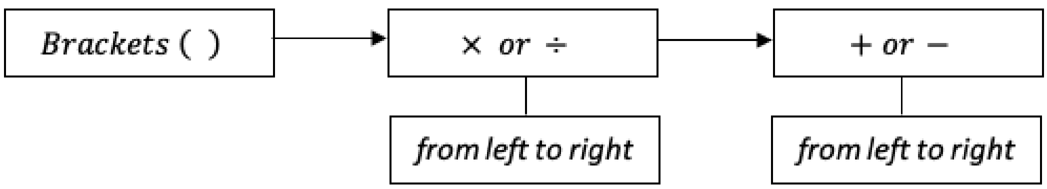
Learning Standards of the Order of Operations (MOE, 2016)

Content standard		Learning standards
1.2 Basic arithmetic operations involving integers	1.2.3	Perform computations involving combined basic arithmetic operations of integers by following the order of operations.
	1.2.4	Describe the laws of arithmetic operations which are Identity Law, Communicative Law, Associative Law and Distributive Law.
	1.2.6	Solve problems involving integers based on the order of operations

The proficiency in comprehending the order of mathematical operations is pertinent to Malaysian high school students' mathematics learning. The presentation of the order of operations in Grade 7 Malaysian mathematics textbook illustrates how the hierarchy is typically taught (see **Figure *2.3***).

Figure 2.3

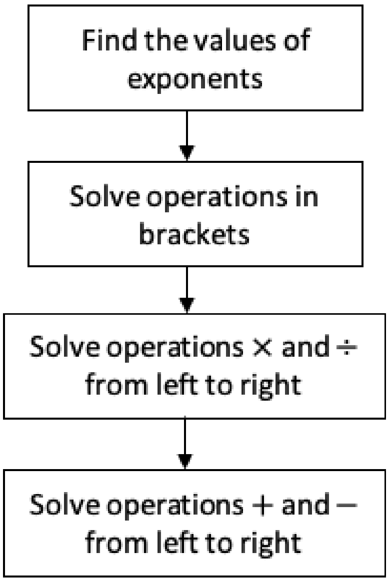
The Order of Operations in Malaysian Grade 7 Textbook (Adapted From Ooi et al., 2016, p. 10)



As illustrated in **Figure *2.3***, at this stage, the order of operations is typically taught without including operations of exponents since the students have not learnt exponents yet. Seventh graders, at this stage, are expected to perform computations involving combined basic arithmetic operations by following the order of operations (MOE, 2016). When they learn exponents, the hierarchy are again illustrated in the "smart tips" of this textbook, encompassing operations on exponents (see **Figure *2.4***).

Figure 2.4

The Ordering in Performing Mathematical Operations in Malaysian Grade 7 Textbook (Adapted From Ooi et al., 2016, p. 68)



At this point, students should perform calculations involving the combined basic operations and operations on exponents based on the order of operations (MOE, 2016).

However, there appears a difference between the hierarchy found in the textbook (**Figure *2.4***) and the typical order convention. Specifically, the typical order convention indicates that operations within parentheses must be evaluated first, but according to the hierarchy in the textbook, students are encouraged to perform operations on exponents first. This hierarchy may have some limitations in describing the complex order of operations in simple rules. Take the expression $(12 - 4)^2$ for example, the subtraction operation, which is in the parentheses, must be executed first before finding the value of the exponent. Although a mathematics textbook is not a government-mandated material in Malaysia, the textbook is a primary source and regularly used for mathematics teaching (Julie & Maat, 2021). This is not unexpected because a textbook is structured in such a way closely following the mathematics syllabus outlining all the learning areas that must be covered and suggesting activities relevant for teaching. It provides a good reference for teachers to complete the syllabus without overlooking some contents (Lepik, 2015; Tarr et al., 2008). In view of the textbook may be the primary reference for teaching the order of operations, it is questionable how expressions are evaluated by Malaysian PSTs who are going to teach the topic in the near future.

2.2.1 The Viral Expressions

The topic of order of operations has sparked a heated debate across the internet. The mathematical expressions that became the point of debate were, $8 \div 2(2+2)$ and $6 \div 2(1+2)$ (Asked & Answered, 2020). These seemingly simple expressions have had different individuals respond differently, but neither one accepts the other's answer.

The order of operations has driven some teachers to argue that such expressions must be evaluated from left to right after simplifying the addition within the parentheses (e.g., Linkletter, 2019; Talwalker, 2019). For example,

$$
\begin{aligned}
8 \div 2(2+2) &= 8 \div 2(4) \\
&= 4 \times 4 \\
&= 16
\end{aligned}
$$

Since multiplication and division have the same precedence, calculations must be performed in order of appearance.

On the other hand, the idea of implicit multiplication has driven other teachers to claim that multiplication should be performed first after calculating the addition within the parentheses (e.g., Asked & Answered, 2020; Peterson, 2019; Talwalker, 2019). For example,

$$
\begin{aligned}
8 \div 2(2+2) &= 8 \div 2(4) \\
&= 8 \div 8 \\
&= 1
\end{aligned}
$$

These teachers presumed implicit multiplication holds a tight bond giving rise to a higher priority over division.

In fact, there still exist ambiguities in mathematical expressions regarding order of operations, particularly when pronumerals and vinculum are involved. For example, mathematicians, scientists, and engineers are likely to accept that $\frac{ab}{cd}$ is mathematically the same as ab/cd. However, some might view the expressions differently especially when the expressions are needed to be evaluated strictly following the left-to-right order for multiplication and division.

Despite this social media controversy, there appears no former research publication about this problem (Linkletter, 2019). Although it may be that the viral expressions were twisted purposely to be ambiguous in order to provoke discussions that facilitate the understanding about the order of operations, some researchers have argued that there are no underlying abstract principles of mathematics for these two expressions (Chang, 2019). The expressions could be well-defined with a clearer syntax using parentheses, such as $(8 \div 2)(2+2)\ or\ 8 \div (2(2+2))$, so that each expression always yields the same answer. This is why the understanding about order of operations is important, everyone should adhere to this convention for consistency and accuracy in evaluating mathematical expressions.

Some researchers suggested that a problem in context can be set to substantially avoid the ambiguity (e.g., Bay-Williams & Martinie, 2015; Cardone, 2015; Jeon, 2012). Consider the following contextualised problem as an example.

> *A table of eight plates where half is red and another half is yellow, each plate contains 2 apples and 2 oranges. How many fruits are there in all red plates?*

Given the context, translating the problem into an expression yields $(8 \div 2)(2 + 2)$, which will not give rise to different interpretations. However, limited studies have investigated the use of contextualised problems in relation to the order of operations. The present study seeks to explore how PSTs determine the order of operations of contextualised problems.

2.2.2 Pre-Service Teachers' Difficulties in the Order of Operations

The order of operations may appear random since some individuals consider the rules as arbitrary conventions and view them as a matter of procedural knowledge (Papadopoulos, 2015). To address these seemingly arbitrary conventions, acronyms, such as PEMDAS and BODMAS, and mnemonic, such as Please Excuse My Dear Aunt Sally, are used to remember the rules (Zazkis & Marmur, 2018). PEMDAS, which is commonly used in the United States (US), stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. The P (Parentheses) of the acronym is replaced with B (Brackets) to become BEDMAS that is used in Canada. The E (Exponents) of the acronym is replaced with O (Orders) to become BODMAS that is widely recognised in Australia. The acronym BODMAS also appears in the Malaysian Grade 8 mathematics textbook suggesting combined operations must be solved based on this acronym (Baharam et al., 2017, p. 24). Other than BODMAS, some Malaysian teachers use KUBADATATO (Kurungan, Bahagi, Darab, Tambah, Tolak) that is translated into brackets, divide, times, plus, minus to aid in the memorisation of the order of operations.

Despite the use of acronyms intending to help students remember the rules, researchers have found that the acronyms create misunderstandings and cause errors in the order of operations (e.g., Dupree, 2016; Glidden, 2008; Pappanastos et al., 2002). One error is related to evaluating expressions with multiplication and division. Glidden (2008) reported that approximately 31% of 381 the US pre-service elementary teachers performed multiplication before division for the expression $24 \div 2 \times 3$. These participants interpreted PEMDAS in the order the letters were presented leading them to prioritise multiplication over division. As the letter M comes before the letter D in PEMDAS, the participants misinterpreted MD to mean multiplication takes precedence over division. Another study by Zazkis and Rouleau (2018) also found that 22% of 22 Canadian pre-service elementary teachers made this type of error. These participants calculated division before multiplication because they assumed DM of the acronym BEDMAS meant division takes precedence over multiplication. Instead of calculating from left to right, the PSTs in both studies interpreted acronyms literally, causing them to erroneously prioritise one operation over another.

Glidden (2008) also observed several other errors. Instead of evaluating from left to right for $9 - 4 + 3$, approximately 31% of the PSTs performed addition first. This error may be because the participants misinterpreted the letters AS of PEMDAS meant addition precedes subtraction. Instead of doing multiplication before addition for $3 + 4 \times 2$, around 22% PSTs calculated addition first. A potential reason for prioritising multiplication over addition is that the PSTs presumed operations must be performed from left to right (Blando et al., 1989; Dupree, 2016; Herscovics & Linchevski, 1994). Furthermore, Glidden (2008) also reported that 80% of their participants obtained an erroneous answer 9 for the expression -3^2. Glidden's (2018) study, however, lacked the data to show the root causes of the errors. The present study attempts to address this gap by exploring the reasons for such errors in the event that PSTs make the errors.

The literature does not appear to critically discuss difficulties of evaluating exponents and parentheses in relation to the order of operations. As argued by Lee and Messner (2000), evaluating stacked exponents may give inconsistent answers. Using the stacked exponents 2^{3^2} as an example, she claimed that one might find the value of 2^3 first instead of doing from right to left. When there is no indication of grouping for stacked exponents, it raises questions of ordering. Given that the ordering of stacked exponents has given limited attention in relation to the order of operations, the present study seeks to provide empirical data to address this gap.

Research on the use of parentheses in relation to the order of operations is relatively thin and has largely focused on how students perform calculations when presented with expressions (Gunnarsson & Karlsson, 2014; Gunnarsson et al., 2016; Papadopoulos & Gunnarsson, 2018, 2020). Gunnarsson et al. (2016) analysed if the use of emphasising brackets will enhance Swedish students' learning of the order of operations. In this sense, emphasising brackets is designed to highlight the structure of an expression. For example, emphasising brackets in the expression $(6351 \div 3) \times 14$ is used to highlight that $6351 \div 3$ is to be computed first. In fact, this expression is mathematically the same as $6351 \div 3 \times 14$ because it is understood that the division operation must be conducted first. Another study by Papadopoulos and Gunnarsson (2018) revealed that primary school students used mental brackets to decide the order of evaluating expressions. Other studies analysed the relation between emphasising brackets and structure sense perceived by students (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011). While it is believed that analysing students' use of parentheses in performing calculations is beneficial, I claim that such work should be based on an understanding on how PSTs use parentheses in relation to the order of operations. However, limited studies were conducted in this study and therefore, the present study aims to fill this gap.

In Malaysia, research examining the order of operations has focused on students' comprehension of the rules. The literature demonstrates that Malaysian students were not completely sure how to apply the order of operations (e.g., Bakar, 2020; Lim, 2010; Singh et al., 2016; Ting et al., 2017). For example, Lim (2010) found that students evaluated expressions largely from left to right, without giving priority to any operations. Bakar (2020) and Singh et al. (2016) reported that students were confused and had difficulties in using BODMAS. However, there is no research on the understanding of Malaysian PSTs in this topic. Therefore, the present study explores if PSTs encounter difficulties in the order of operations and investigates the reasons of the difficulties.

2.3 MATHEMATICAL CONNECTIONS IN RELATION TO THE ORDER OF OPERATIONS

Defined in a general sense, a *mathematical connection* is a relationship between mathematical concepts. In addition to this general definition, some national standards and researchers have suggested dividing mathematical connections into *intra*- and *extra-mathematical connections* (e.g., Gamboa et al., 2021; NCTM, 2000). The former concerns relationships between "representations, definitions, concepts, procedures and propositions within the context of mathematics" (Gamboa et al., 2021, p. 4). The latter refers to relationships between mathematics and contexts outside mathematics. The exploration of connections of the present study is in relation to the former (hereafter referred to as mathematical connections).

Many national curricula have highlighted the importance of making mathematical connections (e.g., ACARA, 2018; Department for Education England [DfE], 2021; MOE, 2016). Despite these acknowledgements, making such connections may be problematic for PSTs because the practices require PSTs to be able to recognise the different connections across mathematics concepts before making them explicit and comprehensible to students (Ponte & Chapman, 2008; Zazkis & Rouleau, 2018). This study focuses on the mathematical connections between the order of operations and the properties of operations because it is a recognised challenge about how PSTs make sense of the seemingly arbitrary conventions. Although the order of conventions might be considered as arbitrary, the conventions are indeed guided by certain conventions. Thus, making such connections with the order of operations is important for PSTs so that they can help their students to make sense of the reasons for the conventions.

2.3.1 Making Connections to the Properties of Operations

Making mathematical connections is a process of linking two or more mathematical concepts. To understand the connections within mathematics, researchers have proposed different categories to describe the connections (Businskas, 2008; Eli et al., 2011; Evitts, 2004; García-García & Dolores-Flores, 2021; Hatisaru, 2022; Rodríguez-Nieto et al., 2022). **Table 2.2** provides descriptions on each of the connection types. Although the studies on mathematical connections used different names for their connection types, their categorisations show some common aspects. In the following paragraphs, the similarities are discussed in relation to making connections between the order of operations and the properties of operations.

In the investigation of connections PSTs made when solving mathematical problems, Evitts (2004) suggested five distinct connection types as presented in **Table 2.2**. His study shared two similar categories with Rodríguez-Nieto et al. (2022). First, Evitts's (2004) structural connections and Rodríguez-Nieto et al.'s (2022) feature/ property connections are similar in terms of comparing the characteristics of related concepts. Based on this connection type, the order of operations and the properties of operations can be linked because both ideas are used when simplifying expressions and will yield the same final answer. A connection can also be established when viewing the distributive property as complementary the order of operations.

Table 2.2

Descriptions of Mathematical Connection Type

Studies	Connection type	Brief description
Evitts (2004)	Modelling	Interactions between mathematics and real-world.
	Structural	Similarities of mathematical concepts.
	Representational	Connections between different representations of a concept.
	Procedure-concept	Links of conceptual knowledge and procedural knowledge.
	Strands of mathematics	Relations among mathematical domains.
Eli et al. (2011)	Categorical	Description based on general features.
	Procedural	Use of mathematical procedures and examples.
	Characteristics/ property	Definition or description based on properties of a concept.
	Deviation	Construction of new concept based on another concept.
	Curricular	Consideration of the impact on mathematics curricular.
Rodríguez-Nieto et al. (2022)	Part-whole	Inclusion and generalisation.
	Feature/ property	Similarities and differences in terms of characteristics or properties of a concept.
	Analogical	Connections between a familiar situation and a concept.
	Different representations	Equivalent and alternate representations.
	Implication/ if-then	Use of a logical relationship to link a concept to another.
	Instruction-oriented	Incorporations into mathematics teaching and learning.
	Procedural	Utilisation of rules, formulae, algorithms.

This is because the distributive property is a way to rewrite expressions without changing their meaning and the order of operations is a way to evaluate expressions. However, there is an argument that the distributive property contradicts the order of operations because the order conventions allow solving whatever operations in parentheses first, but the distributive property allows multiplying each term inside the parentheses by the term outside (Peterson, 2005). In light of this connection, the present study seeks to understand how PSTs recognise and make sense of the connection between the order of operations and the distributive property.

Second, both Evitts (2004) and Rodríguez-Nieto et al. (2022) also included the representational aspect in their categorisations to refer to the connection that was made when a concept was transformed into different representations graphically, numerically, symbolically, or verbally. As an extension, Rodríguez-Nieto et al. (2022) further divided the representations connections into equivalent and alternate representations. An equivalent representation refers to the transformation of different representations made in the same representation system, such as from a graphical form to a graphical system, whereas an alternative representation refers to the transformation of different representations made across different representation systems, such as from graphical to symbolic. When making connections in relation to the order of operations, acronyms and mnemonics used to memorise the procedures are often referred to (Dupree, 2016; Zazkis, 2018). Another representation is the hierarchical triangle indicating the order of different operations. These are considered in making alternative representation connections to the order of operations.

Eli et al.'s (2011) categorisation also showed some similarities with Rodríguez-Nieto et al.'s (2022). For example, both these studies included a category to describe the connection made when a new concept was building from another concept. In Eli et al.'s (2011) study, they termed this type as deviation whereas Rodríguez-Nieto et al. (2022) grouped it as an inclusion aspect under the part-whole connection type. The part-whole connection type is demonstrated in two ways, namely, inclusion and generalisation. Inclusion occurs when a concept is contained in another concept and generalisation is present when a mathematical idea is being generalised. This connection type could be used when making sense of the left-to-right order using the associative and inverse properties. Take the expression $a - b + c$ as an illustration.

$(a-b)+c=[a+(-b)]+c$	subtraction is an inverse operation of addition
$=a+[(-b)+c]$	associative property of addition
$=a+(-b)+c$	
$\neq a-(b+c)$	

In the above expression, $-b$ is viewed as $+(-b)$ because subtraction is an inverse operation of addition. Based on the associative property of addition, the terms in the expression can be regrouped to become $a+[(-b)+c]$ but this is not equal to $a-(b+c)$. This shows that performing operations from left to right is not the same as doing from right to left. Therefore, the left-to-right order is necessary.

In another case, take an expression $a+b-c$ as an example.

$(a+b)-c=(a+b)+(-c)$	subtraction is an inverse operation of addition
$=a+[b+(-c)]$	associative property of addition
$=a+(b-c)$	

Subtraction is an inverse operation of addition so $-c$ is viewed as $+(-c)$. The associative property of addition allows the terms in the expression to be regrouped. This example illustrates that when the expression is in the form of $a+b-c$, following the left-to-right order or performing out of order yield the same answer. Concerning the usefulness of properties of operations in explaining why the left-to-right order works, the present study aimed to investigate the extent to which PSTs can make such connections.

Many categorisations included procedural connections as one of the connection types (Businskas, 2008; Eli et al., 2011; García-García & Dolores-Flores, 2021; Rodríguez-Nieto et al., 2022). One potential explanation for this inclusion is that procedural knowledge is the knowledge needed to demonstrate the fluency in completing steps and algorithms in mathematics. Using rules and formulae in solving mathematical problems and constructing examples to relate mathematical ideas are the traces of making mathematical procedural connections (Businskas, 2008; Eli et al., 2011; Rodríguez-Nieto et al., 2022). The order of operations itself contains rules that reflect seemingly arbitrary conventions about what operation to perform first. Connections are made when the procedures are followed, and examples are constructed.

Despite several researchers discussing mathematical connections in different fields, for example, functions (García-García & Dolores-Flores, 2021; Hatisaru, 2022), geometry (Eli et al., 2013), and measurement (Gamboa et al., 2020), research on making mathematical connections in relation to the order of operations has been rare. One study on this area was conducted by Zazkis and Rouleau (2018). They required 15 Canadian pre-service elementary teachers to discuss about the idea of doing division before multiplication. They found that 70% of the participants agreed with the idea due to the mnemonic BEDMAS but failed to use associativity to explain the order of operations. Although Zazkis and Rouleau (2018) examined the use of properties of operations in explaining the left-to-right order, they focused on the operations of multiplication and division. As an extension of their work, the present study examined the operations of addition and subtraction in the calculation and investigated the connections between the order conventions and the distributive property.

2.3.2 Making Mathematical Connections as a Component of Mathematical Pedagogical Content Knowledge

Making connections across mathematical concepts is imperative because it supports student learning (Coles & Sinclair, 2019; Toh & Choy, 2021). In this regard, PSTs need to possess a deep understanding of mathematics before they can help their students make such connections. Considering that making mathematical connections is linked to PSTs' knowledge of content (Askew et al., 1997; Ball et al., 2008; Hughes, 2016; Rowland, 2013), this study focuses on how PSTs make the connections as an aspect of mathematical pedagogical content knowledge.

Several teacher framework researchers have argued strenuously for the significance of mathematical connections (e.g., Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Rowland, 2013). In Mathematical Knowledge for Teaching (MKT), for example, Ball et al. (2008) highlighted the awareness of how mathematical topics were interrelated as an indication of making connections. In Knowledge Quartet (KQ), Rowland (2013) suggested a connection dimension to describe mathematics teaching development. In addition to sequencing of mathematical topics, Rowland's (2013) connection dimension included the ordering of mathematical tasks. Another study conducted to understand the knowledge, beliefs, and practices used in teaching numeracy claimed that an effective teacher acquires "a rich network of connections between different mathematical ideas" (Askew et al., 1997, p. 4). These studies used the connection construct as one of the key aspects to understand coherency of mathematics teaching. These teacher knowledge frameworks are discussed in detail in Chapter 3 (p. 47).

2.4 CONTEXTUALISED PROBLEMS

This section addresses literature associated with contextualised problems implemented in mathematics education. In particular, I discuss the meanings and types of contextualised problems. Past studies on PSTs in this area are also discussed.

2.4.1 Defining Contextualised Problems

Many students have negative attitudes towards the study of mathematics (e.g., Clarke & Roche, 2018; Reinke, 2019). They often see mathematics as boring and irrelevant (Van den Heuvel-Panhuizen, 2005). To address this issue, many research studies suggest using contextualised problems in teaching mathematics (e.g., Clarke & Roche, 2018; Reinke, 2019). In a broad sense, a *contextualised problem* refers to a problem that is imaginable and experientially real either in the minds or from real experience of an individual.

In fact, the recommendation of using contextualised problems in teaching mathematics has a long history in Dutch mathematics. The Realistic Mathematics Education (RME) recommends working on contextualised problems to develop students' understanding (Freudenthal, 1991, as cited in Van den Heuvel-Panhuizen & Drijvers, 2014). In this regard, the contextualised problems proposed by RME are realistic situations in which the context of the problems is imaginable for students (Van den Heuvel-Panhuizen, 2005). According to Van den Heuvel-Panhuizen & Drijvers, (2014), the proposed realistic situation serves as

> a source for initiating the development of mathematical concepts, tools, and procedures and as a context in which students can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general and less context specific (p. 521).

From this view, the reality principle emphasised in RME aims not only to develop students' ability to apply mathematics in solving real-life problems, but also encourages the use of contextualised problems as a starting point to develop mathematical concepts and ideas.

Contextualised problems play a prominent role in connecting classroom instructions to the world outside the classroom. Using contextualised problems in mathematics teaching may scaffold students' understanding as the problems position the learning of mathematics in realistic settings. It is arguable that contextualised problems may also provide an opportunity for students to discover how mathematics helps them to make sense of the world (Clarke & Roche, 2018; Meyer et al., 2001). Acknowledging the potential of contextualised problems, many national curricula, such as NCTM (2000, 2009) and ACARA (2018), support the use of contexts in teaching mathematics.

To better understand contextualised problems, interpreting what *context* means is necessary. There are a number of views on what a context means in the mathematics education literature. Van den Heuvel-Panhuizen (2005), for example, justified two different meanings of the term context. First, the term represents a learning environment, which includes situations when learning takes place and the interpersonal dimension of learning. Second, it indicates the characteristics of a problem used in a mathematics classroom. As highlighted in the theory of RME, one of the characteristics of contextualised problems is its potential to be presented as entry points for new mathematical concepts (Dapueto & Parenti, 1999; Van den Heuvel-Panhuizen, 2005). Another characteristic of contextualised problems is attributed to real-life situations (Harvey & Averill, 2012; Wijaya et al., 2015). In this case, the notion real life has a similarity with RME's interpretation of realistic in which it is being associated with students' real and imagined experiences (Lee, 2012). Although some teachers may refer a word problem as a contextualised problem (Chapman, 2003), I argue that a word problem is not necessary always a contextualised problem.

For example, the word problem *What is the sum of 8 and 5?* should not be recognised as a contextualised problem because no context is included in the problem.

In view of the above descriptions, this study uses the term contextualised problems instead of word problems. Taken as a whole, contextualised problems in this study indicate mathematical problems that reflect realistic situations, real or imaginable, and may serve as a source for learning the order of operations.

2.4.2 Types and Characteristics of Contexts

De Lange (1995) distinguishes three categories of contexts in mathematical problems. First, problems may have no context. The problems refer directly to mathematical objects, symbols, or structures (Wijaya et al., 2015). For example, the problem *What is 10% of* 5^2*?* would be considered as no context.

Second, problems may have camouflage context in which the problems are "dressed up" (Harvey & Averill, 2012, p. 42) with realistic contexts. This is similar to what Gainsburg (2008) called "student-solved word problem" (p. 204). An example of a problem with camouflage context is *Determine the slope of a hill if the hill is 90m in height and 56m in horizontal distance.* This example represents a camouflage context because students do not need to grapple with the context when solving the problem. The intention of this problem is indicated explicitly by the word slope therefore students can easily follow the formula of slope to get the answer.

The third category is essential and relevant context, in which common sense reasoning is required to solve the problems. Although Wijaya et al. (2015) claims that problems in essential and relevant contexts involve mathematical modelling, the study of de Lange (1995) illustrated that a simple multiple-choice problem can also be considered of having essential and relevant context. The example given in the study of de Lange (1995) is, *"Which of these is the best estimate for the length of a teacher's desk? 4 feet, 10 inches, 2 feet, 15 feet, 20 feet?*" (p. 17). In view that there are different types of contextualised problems, the present study investigates how PSTs determine the order of operations of these several types of contextualised problems.

2.4.3 Studies of Pre-Service Teachers and Contextualised Problems

An analysis of the literature reveals that some research studies examined the nature of the contextualised problems used in a mathematics classroom (Wijaya et al., 2015), some investigated PSTs' ability in writing contextualised problems from experiences in workplace (Nicol, 2002), and some analysed if PSTs can pose contextualised problems from mathematical expressions (Cardone, 2015; Jeon, 2012, Lee, 2012). The study of Wijaya et al. (2015) investigated resources available in Indonesia that were used to support the implementation of contextualised problems in mathematics classrooms. They found that contextualised problems were seldom available in textbooks and of the few contextualised problems available, the contexts were irrelevant and not essential. Examples used in textbooks and classrooms often employ a surface level approach and the use of contexts does not emphasise the development of mathematical thinking (Doorman et al., 2007).

In an attempt to explore the ways in which PSTs incorporated workplace contexts in designing instructional activities, Nicol (2002) reported that their PSTs were less able to design problems with an appropriate context. She found that the mathematics used in the workplace settings were not realised and exploited by the PSTs. For example, the PST participants prepared an activity that required students to label different measurements on a ruler, but this activity did not reflect the context of a problem. Although providing an opportunity for PSTs to experience different workplace settings seems promising, the choice of workplaces needs more careful considerations so that PSTs can identify and focus on the mathematics used in the settings.

Existing research on exploring the ability in writing a contextualised problem from a mathematical expression found that PSTs also encountered difficulties in maintaining the contextual link when designing the problems (e.g., Jeon, 2012; Lee, 2012). Jeon (2012), for example, found that many of her participants created problems that reflected $(5 + 8) \times 6$ even though they were required to create a problem for $5 + 8 \times 6$. It may be that the participants could write a contextualised problem for the expression $5 + 8 \times 6$ but it appears the presentation of the ordering posed a challenge to them. Acknowledging that it is challenging to represent every expression using a real-life example, further research in understanding how context is dealt with to determine the correct order of operation is needed.

Despite many mathematics education communities encourage the use of contextualised problems in classrooms, existing research reported infrequent connections between mathematics instruction and real-world contexts (e.g., Banilower et al., 2013; Contreras & Martínez-Cruz, 2001; Schmidt, 2011). A survey by Banilower et al. (2013) conducted in the US found that 42% of middle school and 29% of high school teachers emphasised the learning about real-life applications when teaching mathematics. This finding is similar with both the studies of Contreras and Martínez-Cruz (2001) and Verschaffel et al. (1997) where their PST participants were also tended to exclude real world connections from appreciations of students' solutions. One of the possible reasons for the infrequent use of contextualised problems is that both in-service and PSTs failed to identify the underlying structure of the contexts (Nicol, 2002; Schmidt, 2011). Thus, it is necessary to explore the extent to which PSTs understand the context of a problem and determine the order of operations based on the context. The potential of contextualised problems to help in learning and applying the order of operations is discussed in Section 2.6.1 (p. 38).

2.5 INTERPRETING STUDENTS' WRITTEN WORK

Interpreting students' written work is one of the essential practices embedded in the teaching of mathematics (Baldinger, 2020). It involves coordinating the mathematical details identified in the students' written work with PSTs' understanding of a mathematical concept (Ivars et al., 2020). Particularly, interpreting students' written work goes beyond analysing the mathematical procedures and identifying the correctness of the work. It also includes explaining students' mathematical thinking underpinning the written work (Battista, 2017).

2.5.1 Students' Errors and Misinterpretation in the Order of Operations

Often, the terms *mistakes*, *errors*, and *misconceptions* are used interchangeably (Gardee & Brodie, 2015). These terms, however, mean differently. *Mistakes* or what Oliver (1989) called "slips" (p. 3) are made due to carelessness. *Errors*, on the other hand, are consequence of misconceptions (Gardee & Brodie, 2015; Sarwadi & Shahrill, 2014; Wischgoll et al., 2015). *Misconceptions* are underlying wrong beliefs and principles that lead to persistent errors (Makonye, 2012). A misconception is formed when a preconception is wrongly understood (Ashlock, 2002). It is "not wrong thinking but is a concept in embryo or a local generalisation that the pupil has made. It may in fact be a natural stage of development" (Swan, 2001, p. 154). Making errors are normal in the process of learning (Brodie, 2014; Heinze & Reiss, 2007; Ingram et al., 2013). Since errors reflect students' understanding of a concept, the occurrence of errors provides an opportunity for teachers to confront and address students' misconceptions.

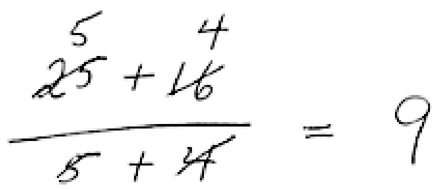
Previous research has shown that students of different grade levels made errors with the order of operations, no matter whether they were in primary school (Herscovics & Linchevski, 1994), secondary school (Blando et al., 1989) or at the college level (Pappanastos et al., 2002), they failed to evaluate mathematical expressions based on the order of operations. As evidenced in the study of Herscovics and Linchevski (1994), 77% of the primary school student participants gave 110 as the answer for the expression $5 + 6 \times 10$. These students had first performed addition then multiplication. A similar finding was also found by Blando et al. (1989) who examined secondary school students' arithmetic errors. In this respect, there appears two possible misinterpretations that constitute to this error. First, the students might believe that addition takes precedence over multiplication; second, they might think that all operations are to be done from left to right. The second misinterpretation was also reported in the study of Pappanastos et al. (2002) where 33.3% of the college students executed $10 + 5 \div 5$ from left to right.

In fact, Pappanastos et al.'s (2002) study on more than 300 college students also reported that 14.8% of the students incorrectly evaluated the expression $6 \div 3 \times 2$. These students calculated 3×2 to get 6, then computed $6 \div 6$ to arrive at the erroneous answer 1. Although no data could indicate what had caused this error, Pappanastos and colleagues anticipated the misconception underlying this error was that the students gave priority to multiplication over division. This error might be due to students misinterpreted the acronym PEMDAS in the order the letters are presented (Cardone, 2015). Another remarkable finding of Pappanastos et al.'s (2002) study is 88.9% of the students incorrectly stated that the expression -5^2 equals 25. In this case, the students might think that the minus sign is attached to the 5, rather -1×5^2.

Another study discussed students' errors was performed by Ameis (2011). He displayed two examples of errors students would arrive at erroneous answers. The first example is related to addition and division (see **Figure *2.5***). In simplifying the expression $\frac{25+16}{5+4}$, the student made error due to a misconception related to a fraction bar. The student was not aware that the fraction bar was actually a grouping symbol. As a result, the student simply followed the order of operations – division must be performed before addition.

Figure 2.5

First Example of Student Error (Ameis, 2011, p. 416)



The second example is related to addition and operating on a radical (see **Figure *2.6***).

Figure 2.6

Second Example of Student Error (Ameis, 2011, p. 416)

In evaluating the expression $\sqrt{9+4}$, the student was likely to have a misconception about a radical. The student was not able to see the radical as a grouping symbol and thus computed the radical before addition.

Errors in the order of operations are often seen in classrooms (Zazkis, 2018). PSTs need to be aware of the error and its underlying reason so that they can support students' learning. This study thus explores how PSTs interpret students' written work in the order of operations at the same time investigates what teaching approaches they will use to avoid students' errors and misinterpretations. Appropriate pedagogical approaches have to be taken otherwise students' errors will persist for a long time (Sarwadi & Shahrill, 2014). Various pedagogical ideas to teach the order of operations are discussed in Section 2.6.1 (p. 38).

2.5.2 Existing Research on Analysing Students' Written Work

Prior research in relation to analysing students' written work often restricted to describing students' procedures and checking students' answers for accuracy (Even & Tirosh, 1995; Gökkurt et al., 2013; Kiliç, 2011; Şahin et al., 2016; Shin, 2020; Tanisli & Kose, 2013; Tirosh, 2000). For example, Kiliç's (2011) presented exemplary student solutions to six pre-service secondary mathematics teachers and required them to check the correctness of the solutions. Similarly, Tanisli and Kose (2013) used exemplary student responses to examine 130 pre-service primary mathematics teachers' abilities in identifying students' errors and predicting students' incorrect answers in relation to variables, equality, and equation. Incorrect exemplary student written work were also presented to 98 prospective secondary mathematics teachers in the study of Şahin et al. (2016) to analyse if the participants could identify the students' errors. These studies found that the participants experienced difficulties in determining students' errors at the same time the participants also had misconceptions in relation to the related mathematical concepts.

Other studies in relation to examining students' written work emphasised on how interventions would support PSTs' interpretations towards students' work (e.g., Ergene & Bostan, 2022; Ivars et al., 2020; Monson et al., 2020; Sánchez-Matamoros et al., 2015). For example, Ivars et al. (2020) examined the learning trajectory of fractions in helping 95 pre-service primary teachers to analyse students' responses on a fraction task. Another study conducted by Ergene and Bostan (2022) investigated how three PSTs responded to students' written solutions before and after the implementation of an intervention in relation to length measurement. These studies showed that interventions potentially strengthen PSTs' analysis of students' written work.

It is believed that much can be gained from helping PSTs analyse students' written work; I claim that such work should be based on an understanding of the process through which PSTs analyse students' written work and interpret students' sense making. Limited studies were conducted to understand how PSTs analyse and interpret students' sense making through written work (Baldinger, 2020). Knowing the ways PSTs used to analyse students' written work is crucial because key features for supporting PSTs to develop abilities in interpreting students' sense making can be identified. Building on this perception, Baldinger (2020) interviewed eight pre-service secondary teachers at the beginning of their teacher training programme and required them to reason about students' understanding of algebra and geometry tasks. She classified the participants' reasoning into three different categories, namely, mathematical, pedagogical, and self-comparison. Note that these categories are discussed in detail in the next section. Baldinger (2020) found that majority of the participants interpreted students' written work mathematically because these participants had limited opportunities to learn about students' common errors or students' mathematical thinking since they were at the beginning of their teacher training programme. Considering PSTs towards the end of their teacher training programme may have more knowledge and experience about what students in general might do and think and how a mathematical concept in general might be taught, the current study extends Baldinger's (2020) work by investigating how PSTs towards the end of their teacher training programme reflect on students' sense making through students' written work.

2.5.3 Approaches for Interpreting Students' Written Work

Students work is interpreted through a variety of approaches. Colestock and Sherin (2009) identified five ways for making sense of students' thinking via watching videos of classroom instruction. The five ways are generalisation, comparison, problem solving, perspective-taking, and reflective thinking. In view of written work provide less clues on students' thinking as compared to videos of classroom instruction, Baldinger (2020) suggested other three approaches to interpret students' written work, namely mathematical, pedagogical, and self-comparison. Although some of the approaches used in observing video clips, such as perspective-taking, are not applicable in interpreting written work, the strategies suggested by Colestock and

Sherin (2009) still serve as a strong starting point to analyse how students' work could be made sense of.

In the following paragraphs, I discuss Baldinger's (2020) three approaches that are used to interpret students' written work. The description of the approaches is summarised in **Table 2.3**.

Table 2.3

Approaches to Interpret Students' Written Work (Adapted From Baldinger, 2020).

Approaches	Descriptions	Examples
Mathematical	Draw on existing mathematical knowledge and prior experience to interpret what students might be thinking, or critique students' solutions.	Draw on the understanding about associative and inverse properties to reason the order of operations. Try out students' ways of evaluating expressions and check if students' solutions make mathematical sense.
Pedagogical	Draw on knowledge about what students might commonly do, how a mathematical concept might generally be taught, and how the curriculum might usually be sequenced.	Relate students' ways of evaluating expressions to common students' errors and misconceptions. Draw on the typical way of teaching the order of operations that is using an acronym.
Self-comparison	Compare PSTs' own solution with those of students to see similarities and notice differences.	Include comments indicating comparing, such as "The same way I did" or "I did differently".

First, reasoning about students' written work may occur through a mathematical approach. Similar to Colestock and Sherin's (2009) generalisation approach used for interpreting classroom video clips, the *mathematical* approach involves drawing on PSTs' existing knowledge and prior experience to interpret what students might be thinking. Many research studies have argued that mathematical approach is an effective strategy where PSTs used what they have learned and have encountered previously to reason students' written work (e.g., Chin & Tall, 2012; Duke & Pearson, 2002; Putnam & Borko, 2000; Shulman, 1987). In addition to drawing on priorly learned content knowledge, Baldinger (2020) included the element of critique students' written work as another element of this approach. In this sense, trying out students' ways of solution and checking if the solutions make mathematical sense is considered mathematical. This is supported by Goldsmith and Seago (2011) in which they claimed that computations may be used to make sense of students' reasoning.

Second, interpreting students' written work may occur through a pedagogical approach. This approach is also referred to as instructional by Tiilikainen et al. (2019) in which they situated the approach in an instructional context based on instructional purposes. Specifically, *pedagogical* approach relies on PSTs' pedagogical content knowledge. Through this lens, PSTs draw on what they know about students might commonly do. This includes knowing the students' common errors and misconceptions. PSTs may relate the students written work with how a mathematical concept might generally be taught and how it might usually be sequenced.

Third, making self-comparison is another approach to interpret students' written work. Different researchers named this approach differently. For example, Hodkowski (2018) named it as juxtaposition, and Rasmussen et al. (2020) named it as connecting. Some researchers further distinguished the self-comparison approach into different types (Sapti et al., 2019; Wilson et al., 2011). Sapti et al. (2019), for example, made distinction between comparing work and comparing knowledge. Despite different names, all the researchers described this *self-comparison* approach as PSTs reasoning how their own solutions are similar to or different from those of students. In other words, PSTs contrast their own solutions with students' solutions when reasoning students' written work (Baldinger, 2020). Although Baldinger (2020) asserts that using the self-comparison approach shows promise as it helps in interpreting students' sense making about a task, this approach may display a danger if the PST's own solution is incorrect. For example, a PST who always prioritises division over multiplication is likely to misinterpret a student's solution. Thus, the self-comparison approach can be a promising strategy to make sense of students' written work provided that the PST's knowledge and solution are accurate.

Research examining approaches PSTs used to interpret students' written work is scarce despite interpreting students' written work is a core of the work of teaching (Kavanagh et al., 2020). It is believed that understanding the ways that PSTs typically interpret students' written work is critical if we are to effectively help PSTs understand students' sense making. Specifically, knowing how PSTs analyse students' written work may suggest key approaches that are likely to be successfully understanding students' sense making. In addition, such information could be useful in identifying challenges that PSTs must overcome when interpreting students' written work. In light of the scarcity of such research and the feasibility of Baldinger's (2020) approaches, the present study investigates how PSTs reflect on students' sense making through students' written work involving the order of operations.

2.5.4 Interpreting Students' Written Work as a Component of Mathematical Pedagogical Content Knowledge

Analysing and interpreting students' work is closely related to mathematical pedagogical content knowledge (Ivars et al., 2020; Son, 2013). In this respect, PSTs must be able to analyse the mathematical procedures, identify the correctness, and interpret students' sense making underpinning students' written work. Considering that interpreting students' work is linked to PSTs' knowledge of students' sense making (Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Rowland, 2013; Tatto et al., 2008), this study focuses on how PSTs interpret students' written work as an aspect of mathematical pedagogical content knowledge.

PSTs must understand students' thinking in order to recognise what students need to learn (Simon, 2022). Considering this, teacher knowledge frameworks have highlighted the need to understand students' sense making through interpreting students' work (e.g., Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Rowland, 2013; Tatto et al., 2008). For example, Ball et al. (2008) claims that PSTs depended on their knowledge of content and students to anticipate what students in general might think and do. PSTs need to move beyond determining right or wrong answers; they must understand the conceptual basis of students' work (Ball, 1990). Similarly, Tatto et al. (2008) included the element of analysing and diagnosing students' solutions and arguments in their teacher framework to address how PSTs should teach mathematics. Furthermore, the need of identifying students' errors through interpreting the students' work is also rooted in the foundation dimension of KQ framework (Rowland, 2013). These studies relevantly used the element of interpreting students' work as an aspect to understand the students' procedures and students' sense making underpinning their written work. These teacher knowledge frameworks are discussed in detail in Chapter 3 (p. 47).

2.6 APPROACHES IN TEACHING THE ORDER OF OPERATIONS

Acknowledging the difficulties in the order of operations as discussed in Section 2.2.2 (p. 18), various ways to approach the order of operations have been introduced by past studies (e.g., Ameis, 2011; Bay-Williams & Martinie, 2015; Cardone, 2015; Dupree, 2016; Golembo, 2000; Jeon, 2012; Taff, 2017; Zazkis, 2018). The following sections review some pedagogical ideas and three approaches in teaching the order of operations.

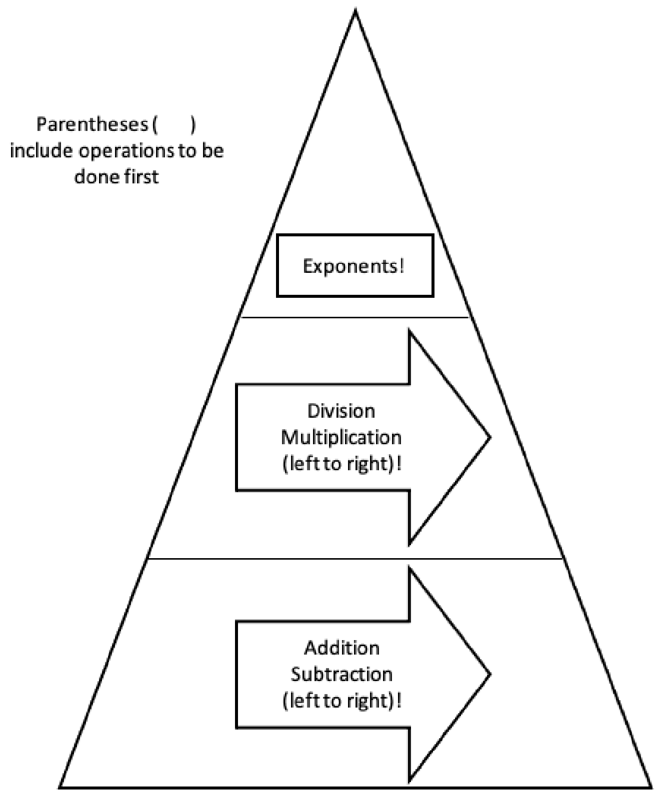
2.6.1 Pedagogical Ideas of the Order of Operations

A well-known approach for teaching the order of operations is using an acronym. As discussed in Section 2.2.2 (p. 18), different acronyms are used in different countries. For example, PEMDAS is prevalent in the US whereas BIDMAS or BODMAS is common in the UK. Acknowledging the unintentional order caused by these acronyms, researchers have introduced new ideas to present the order of operations (e.g., Cardone, 2015; Golembo, 2000; Taff, 2017; Van de Walle et al., 2011). For example, to highlight the operations of the same priority, Golembo (2000) suggests writing the acronym as $BE\frac{D}{M}\frac{A}{S}$. Cardone (2015) proposes using GEMA (Grouping, Exponents, Multiplication, Addition) to indicate all type of grouping symbols such as brackets and fraction bars. Taff (2017) suggests iTAFF (Identify Terms And Factors First) to emphasise the precise use of terms and factors. However, all these ideas are still centred around the use of acronyms to aid the memorisation of the order of operations.

As an alternative to acronyms, researchers have suggested a hierarchical triangle to represent the rules (Ameis, 2011; Bay-Williams & Martinie, 2015). The top priority operation is placed on top of the triangle whereas the less priority operations are placed below (see **Figure *2.7***).

Figure 2.7

Hierarchy of the Order of Operations (Adapted From Bay-Williams & Martinie, 2015)



Because of its visual nature, the hierarchical triangle allows students to make interpretation and draw conclusions more easily. Most importantly, the order of operations triangle clearly shows the operations that are of the same priority (i.e., addition and subtraction, multiplication and division). This is evident in Rahman et al.'s (2017) study in which eighth graders in Brunei Darussalam improved their understanding about the order of operations through introducing the order of operations triangle.

Some studies have recommended focusing on the connections between the order of operations and the properties of operations (e.g., Dupree, 2016; Jeon, 2012; Zazkis, 2018). In Indonesia where no mnemonic was used for the teaching of the order of operations, Sari and Ernawati (2019) claimed that students had a better understanding of the topic when the properties of operations were linked. This is because connecting associativity and the order of operations can help in explaining why the left-to-right order works (Dupree, 2016; Zazkis, 2018). When interpreting subtraction as additive inverse and division as multiplicative inverse, some misinterpretations can potentially be removed. The connections between the order of operations and the properties of operations are presented in detail in Section 2.3 (p. 20).

Other studies have suggested using contextualised problems to help learners learn and apply the order of operations (e.g., Bay-Williams & Martinie, 2015; Cardone, 2015; Holm 2021; Jeon, 2012). As discussed in Section 2.4 (p. 26), a contextualised problem refers to a problem that is imaginable and experientially real for learners either in their minds or from real experience. Although researchers have argued that the order of operations will not be interpreted differently if given a contextualised problem rather than a numerical expression, no evidence has been provided to support this argument (e.g., Bay-Williams & Martinie, 2015; Cardone, 2015; Chang, 2019; Jeon, 2012). Instead of requiring participants to write an expression to reflect a problem, Jeon (2012) asked pre-service elementary teachers to describe a problem that could be represented by the expression $5 + 8 \times 6$, but the problems created reflected $(5 + 8) \times 6$ rather than $5 + 8 \times 6$. Similarly, Lee (2012) required pre-service elementary teachers to pose problems that best represent real-life connections, but 42% of the problems created were general statements and did not display real-life connectedness. The problems created by the participants lacked variety in contexts and most of the problems were related to money or time situations.

Moreover, not all expressions can be represented using a real-life example. For example, it is challenging to exemplify $\left(3 - \sqrt{4}\right)^2$ using a real-life problem. Considering the same interpretation of the order of operations can be achieved through the use of contextualised problems and lack of examination on how PSTs write mathematical expressions to represent the order of operations of contextualised problems, the present study attempted to fill this gap.

Different approaches to teach the order of operations were introduced by different researchers and educators. Hewitt (2012), for example, introduced the Grid Algebra Software to teach the topic. Gunnarsson et al. (2016) used emphasising brackets to highlight the ordering when evaluating expressions. Jeon (2012) and Golembo (2000) propose story writing to understand the order of operations. Recently, Chin et al. (2022b) suggests a theory and offers empirical evidence that spoken articulation could assist the sense making of the order of operations. Instead of viewing the rules as arbitrary convention, two mathematics teachers in their study used mental brackets as a meaningful indication of the order of operations. Given the substantial variation in possible approaches for teaching the order of operations, the present study aims to investigate the approaches Malaysian PSTs plan to use in teaching the topic as a component of their mathematics pedagogical content knowledge.

2.6.2 Transmission, Connectionist, and Discovery

Understanding PSTs' belief towards teaching may provide insight into their perceived teaching approaches (Heimlich & Norland, 2002; Koballa et al., 2000). In view of teachers' beliefs reflect teaching approaches, the following paragraphs discuss three approaches of teaching mathematics as introduced by Askew et al. (1997).

Askew et al. (1997) investigated key factors that contributed to effective teaching of numeracy from a sample of 90 teachers. They used transmission, connectionist, and discovery approaches to describe three patterns of teaching and beliefs. A *transmission* approach emphasises verbal teaching more than learning. In this sense, a clear introductory explanation followed by routine exercises is provided. A transmission classroom is primarily based on student individual activity of rote learning and drilling lead by a teacher. Past studies have argued that a transmission teacher tends to teach standard procedures and students practise the procedures to achieve fluency (Calleja et al., 2021; Francome, 2014; Pratt, 2002; Swan, 2006). It is apparent that this approach encourages "instrumental understanding" as proposed by Skemp (1976, p. 21). An example represents this transmission approach is the use of acronym to teach the order of operations. Using BODMAS or its variants promotes rote memorisations and may result in misinterpretations as discussed in Section 2.2.2 (p. 18). This approach may be less effective because the teaching is centred around remembering a collection of facts and standard methods (Askew et al., 1997).

A *connectionist* approach emphasises dialogues between a teacher and students. This approach involves making mathematical connections between related mathematical contents, encouraging negotiations of meanings via discussions, and including realistic problems in the teaching of mathematics. Students' errors are made explicit in order to refine their methods. Existing studies have found that highly effective mathematics teachers are teachers who emphasise mathematical connections and complexities in their teaching (Askew et al., 1997; Britt et al., 2001; Stigler & Hiebert, 2009). An example represents this connectionist approach is making connections between the order of operations and the properties of operations (e.g., Dupree, 2016; Jeon, 2012; Zazkis, 2018). Sari and Ernawati (2019) argues that students had a better understanding of the order of operations when they used associativity and inverse properties to reason why the left-to-right order works. Another example that reflects the connectionist approach is the use of a hierarchical triangle as illustrated in **Figure *2.7*** (p. 39). This hierarchical triangle can be refined by using more abstract symbols when the order of operations is conveyed via a pictorial or visual representation. Using such representations and requiring students to reason analogically is foundational to mathematical learning (English, 2004, 2016).

A *discovery* approach emphasises learning more than teaching. In this respect, a discovery classroom is predominant by practical activities and reflections of students in order to discover their own methods in doing mathematics (Calleja et al., 2021; Tall, 2007). In other words, this discovery approach involves the flexibility in teaching by exploring options and alternatives put forward by students. With this flexibility in teaching, students must be prepared before learning a new mathematical concept due to the time limitation of a lesson (Mason & Johnson-Wilder, 2004). An example representing this discovery approach is using different scientific calculators to make students aware that different orders of calculations may result in different answers. From there, students may use their own ways to find out the correct order of operations. The characteristics of these approaches relevant to the present study are outlined in **Table 2.4**.

Table 2.4

Transmission, Connectionist, and Discovery Approaches Relevant to the Study

Transmission approach
Emphasises verbal teaching more than learning
Uses of a clear introduction followed by routine exercises
Involves student individual activity of rote learning and drilling
Focuses standard procedures
Involves students practise standard procedures to achieve fluency

Connectionist approach
Emphasises dialogues between a teacher and students
Makes mathematical connections between related mathematical contents
Encourages negotiations of meanings via discussions
Uses of contextualised problems
Recognises students' errors and made explicit

Discovery approach
Emphasises learning more than teaching
Employs practical activities
Invites reflections of students to discover their own methods
Involves the flexibility in teaching by exploring students' options and alternatives
Focuses students' preparation before learning a new concept

In view of its potential implications on teachers' quality, some studies used transmission, connectionist, and discovery to analyse professional development programme (Britt et al., 2001; Calleja et al., 2021), some used the approaches to identify effective teachers (Askew et al., 1997; Rejeki & Sugiyantu, 2015; Swan, 2006), and some extend the three approaches into six principles of practice to understand classroom practices (Muir, 2008). The study of Askew et al. (1997) that designed to identify effective teachers of numeracy found that no one teacher used exactly one teaching approach. They claimed that a connectionist teacher was likely to make greater gain in student learning. Similarly, Britt et al.'s (2001) study that analysed the change in teaching practices of 18 teachers after a professional development program found that their participants with sufficient knowledge of content were more likely to use the connectionist approach to draw mathematical connections after the programme. On the other hand, the study of Rejeki and Sugiyantu (2015) examined beliefs about mathematics of 65 Indonesian PSTs who had completed most university courses and teaching practices in schools found that their participants used mainly a transmission approach in teaching mathematics. Considering the usefulness of these approaches in understanding the strategies in teaching mathematics, the present study investigates how PSTs approach the order of operations that involves a collection of facts and standard methods.

2.7 PRE-SERVICE TEACHERS' MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE

Research investigating knowledge needed for teaching mathematics is complex because it involves different interrelated dimensions or components (Even & Tirosh, 2002; Moloney & Clarke, 2010). This section examines past research studies conducted on mathematical pedagogical content knowledge of PSTs. Research has shown that PSTs demonstrate insufficient mathematical pedagogical content knowledge. Particularly, they have difficulties in anticipating students' common misconceptions and identifying students' errors (Even & Tirosh, 1995; Gökkurt et al., 2013; Kiliç, 2011; Şahin et al., 2016; Tanisli & Kose, 2013; Tirosh, 2000). An example is the study of Kiliç (2011) that examined six PSTs' mathematical pedagogical content knowledge in a mathematics methods course. In her study, PST participants were unable to identify students' errors and the source of students' misconceptions in relation to multiplying binomials, rational, polynomial equations, and inequalities. Kiliç (2011) claims that the reason for this failure is the superficial conceptual knowledge of PSTs. The finding of Kiliç's (2011) study is consistent with that of Even and Tirosh (1995). The study of Even and Tirosh (1995) also found that their PST participants had inadequate subject knowledge of the concept functions.

Another study that revealed the inability of pre-service primary mathematics teachers to understand students' thinking was conducted by Tanisli and Kose (2013). They found that the insufficient subject knowledge of PSTs had resulted in their difficulty in explaining students' thinking in relation to algebraic concepts. Not only that, they also observed that their PST participants had misconceptions in algebraic concepts and the PST participants could not recognise their own misconceptions. For example, their PST participants described the algebraic expression $4n + 7$ as an identity but they were unable to define the concept of identity. As a result, their misconceptions prevented them from reasoning about students' thinking process.

Other than having problems in predicting students' misconceptions and determining students' errors, past research has also shown that PSTs fail to generate effective ways of teaching (Gökkurt et al., 2013; Kiliç, 2011; Şahin et al., 2016; Tirosh, 2000). An example is the study performed by Even and Tirosh (1995), in which two students' erroneous answers ($4 \div 0 = 0$ and $4 \div 0 = 4$) were presented to PSTs and they were required to suggest effective ways of teaching. Although the PST participants managed to identify the students' errors, they showed no extra effort to understand students' reasoning behind the errors. As a result, the suggested teaching approaches were simple and less effective.

Similar to Evan and Tirosh's (1995) study, the study of Şahin et al. (2016) on 98 pre-service primary mathematics teachers also showed that their PST participants were unsuccessful in enacting conceptual understanding in their teaching of fractions. The primary teaching method of their PST participants was reminding students to memorise mathematical rules about fractions. The PST participants were not able to synthesize different ways of teaching fractions. Likewise, the study of Delice and Sevimli (2010) reported that their PSTs used not more than one way to solve problems related to integrals.

Taken as a whole, the reviewed literature suggests that mathematical pedagogical content knowledge utilised by PSTs is not at a desired level. PSTs understand the content and can perform operations on it, but they have little understanding of how to apply the knowledge for teaching purposes (Kinach, 2002). As students' learning of mathematics depends fundamentally on teachers' skills and knowledge (Boaler & Humphreys, 2005), an undesired level of mathematical pedagogical content knowledge may negatively affect students' learning.

The components of mathematical pedagogical content knowledge as discussed in Section 3.4 (p. 52) are not independent, rather, they intensively interact with each other (Marks, 1990). This highlights the significance of studying all the components of mathematical pedagogical content knowledge in a research setting. However, past studies often concentrate on one or a few of the knowledge components (Cueto et al., 2017). The study of Şahin et al. (2016), for example, investigated only two components of mathematical pedagogical content knowledge with respect to fractions. They called for future studies on other learning domains of mathematics and also on other components of mathematical pedagogical content knowledge. The present study thus responds to Şahin et al.'s (2016) call and seeks to explore three components of PSTs' mathematical pedagogical content knowledge of the order of operations. The topic of the order of operations is selected because the literature lacks an examination on pedagogical knowledge in relation to the order of operations, and the literature reveals that PSTs have misconceptions of the topic as discussed in Section 2.2.2 (p. 18).

Examining PSTs' mathematical pedagogical content knowledge is likely to contribute to the development of this knowledge in methods courses of PST education (Tirosh, 2000; Van Driel et al., 2002). Such courses can be designed in a way that PSTs would have opportunities to form effective teaching approaches, and to evaluate students' work. Thus, a study in this area is worthy of investigation.

Chapter 3: Conceptual Framework

3.1 CHAPTER INTRODUCTION

This chapter describes the conceptual framework that underpins the analysis of knowledge needed in teaching mathematics. In consideration of a suitable framework for this study, mathematical pedagogical content knowledge is used to gain a greater understanding of PSTs' knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches required to teach the order of operations. This chapter begins by describing existing frameworks used in understanding mathematics teaching (Section 3.2). The conceptual framework of this study is presented in Section 3.3 and key categories of mathematical pedagogical content knowledge are discussed in Section 3.4. The chapter concludes with a brief summary.

3.2 EXISTING CONCEPTUALISATIONS OF MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE

The research on pedagogical content knowledge is firmly rooted in the study of Lee Shulman (1987). Realising an imbalance between pedagogy and content knowledge, he introduced the notion of pedagogical content knowledge as a specific domain of teacher knowledge for teaching (Shulman, 1986, 1987). *Pedagogical content knowledge* refers to "the particular form of content knowledge that embodies the aspect of content more germane to its teachability" (Shulman, 1986, p. 9). According to Hill et al. (2008), however, there is debate regarding Shulman's definition of pedagogical content knowledge as the term lacks a universally agreed-upon definition.

Subsequently, mathematics researchers from different nations around the world have further refined Shulman's pedagogical content knowledge by conceptualising it into various knowledge components (e.g., Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Rowland et al., 2005; Schoenfeld, 2013; Tatto et al., 2008). In England, for instance, Askew et al. (1997) categorised mathematical pedagogical content knowledge into three distinct components: numeracy subject knowledge, knowledge of how pupils learn numeracy, and knowledge of numeracy teaching approaches. Their aim was to demonstrate the interconnectedness of beliefs, knowledge, and practices pertaining to the teaching of numeracy. Rowland et al. (2005) also conducted a study on mathematical pedagogical content knowledge in the UK. By analysing pre-service elementary mathematics teachers' lessons, they created the Knowledge Quartet (KQ) framework that consists of four dimensions: foundation, transformation, connection, and contingency. The contingency dimension of the KQ framework presents the greatest challenge to effective teaching as it emphasises the importance of responding to students' spontaneous ideas.

In the US, Ball et al. (2008) developed Mathematical Knowledge for Teaching (MKT) framework that contains six domains of knowledge: common content knowledge, knowledge at the mathematical horizon, specialised content knowledge; knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Despite the relevance of using the MKT framework in analysing teacher knowledge needed for teaching mathematics, the MKT research team recognised a limitation in terms of describing the knowledge domains. They acknowledged, "the problems of definition and precision exhibited in our current formulation" (Hill et al., 2008, p. 404). Schoenfeld (2013) also conducted a study in the US aimed at assisting mathematics teachers in creating powerful classroom practices. He developed a framework known as Teaching for Robust Understanding (TRU) that comprises five dimensions to describe the extent to which a mathematics classroom is deemed powerful. The five dimensions are (i) the content, (ii) cognitive demand, (iii) equitable access to mathematics, (iv) agency, ownership, and identity, and (v) formative assessment. The framework, however, does not outline how a mathematics teacher can teach effectively (Schoenfeld, 2020). Consequently, rather than focusing on mathematics teachers, the framework places emphasis on students and the classroom environment.

In Australia, Chick et al. (2006) classified a framework for analysing mathematical pedagogical content knowledge into three broad categories. These categories are clearly pedagogical content knowledge, content knowledge in a pedagogical context, and pedagogical knowledge in a content context. Although Chick et al.'s (2006) framework is comprehensive with many elements corresponding to each category, some elements lack specificity because the meaning and significance of the mathematical structure are not explained (Vale et al., 2010).

In terms of cross-national study, Tatto et al. (2008) conducted the Teacher Education Development Study-Mathematics (TEDS-M) project that involves 17 countries (Botswana, Canada, Chile, Chinese Taipei, Georgia, Germany, Malaysia, Norway, Oman, Philippines, Poland, Russian Federation, Singapore, Spain, Switzerland, Thailand, and the US). This project was the first empirical international project of teacher preparation. Unlike other frameworks, Tatto et al.'s (2008) framework classified mathematical pedagogical content knowledge according to three different teaching phases: mathematical curricular knowledge, knowledge of planning for mathematics teaching and learning, enacting mathematics for teaching and learning.

Despite employing various categorisations, these frameworks exhibit shared elements in describing the knowledge needed for teaching mathematics. First, all the frameworks include mathematical content knowledge as a fundamental component within the conceptualisation of mathematical pedagogical content knowledge. Existing teacher knowledge researchers recognise that a teacher's understanding of mathematical contents relevant to the teaching context is of utmost importance. This understanding encompasses not only the teachers' fluency and scope of mathematical content used for teaching but also their ability to make connections within mathematics. For example, Askew et al. (1997) described this knowledge as having two distinct aspects: *knowledge of content* and *knowledge of relationships*. The former concerns "knowledge of facts, skills and concepts of the numeracy curriculum" and the latter refers to "knowledge of how different aspects of mathematics content relate to each other" (Askew et al., 1997, p. 55). In fact, the understanding about mathematical connections is consistent with Richard Skemp's *relational understanding* that describes the knowledge used for linking mathematical ideas in a coherent way and building connected knowledge structure (Tall, 2013).

Second, the aforementioned frameworks also incorporate the knowledge used for understanding students' learning. This knowledge draws attention to students' existing knowledge, areas of complexity or difficulty they may encounter, and potential misconceptions they might hold. In this conceptualisation, the need to identify students' errors and anticipate students' misconceptions about mathematical contents is highlighted. In addition, some frameworks also encompass teachers' ability in responding to students' spontaneous responses observed during classroom instructions (Rowland et al., 2005; Tatto et al., 2008). As previously discussed, Rowland et al. (2005) elucidated that the contingency dimension within the KQ framework is characterised as “almost impossible to plan” (p. 263) due to its reliance on teachers' responses to unforeseen classroom events. It can be argued that the contingency dimension presents the most significant challenge to effective teaching because it requires maintaining a focus on achieving planned learning standards, even when the agenda deviates.

Third, the existing frameworks of mathematical pedagogical content knowledge include the knowledge used to organise and teach mathematics in a manner that makes the content comprehensible and accessible to students. To capture the multi-dimensionality of this knowledge, some teacher framework researchers incorporate an additional aspect beyond cognitive attributes, that is teachers’ beliefs of teaching mathematics (e.g., Askew et al., 1997; Rowland et al. 2005; Tatto et al., 2008). These researchers believe that teachers’ perceived ways of teaching can be identified through understanding their belief orientations towards teaching (Heimlich & Norland, 2002; Koballa et al., 2000). For example, a belief in the importance of being able to perform standard mathematical procedures may lead to heavy reliance on paper and pencil method in teaching. In this respect, Askew et al. (1997) proposed transmission, connectionist, and discovery orientations to examine teachers’ perceived ways to teach numeracy. These orientations are discussed in Section 2.6.2 (p. 41).

The aforementioned research has yielded valuable insights into the components and dimensions of mathematical pedagogical content knowledge. Building upon these existing conceptualisations, the present study aims to delve deeper into the specific context pertaining to the order of operations. Since the focus of this study is on PSTs’ mathematical pedagogical content knowledge, a more specific framework is warranted, which is described in the following section.

3.3 MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE AS THE CONCEPTUAL FRAMEWORK

The framework of mathematical pedagogical content knowledge was specifically designed for this study, drawing upon key elements from other frameworks discussed in the previous section. In general, this framework has the potential to describe and explain the knowledge used for teaching a wide range of mathematical topics and concepts. Specifically, in the current study, this conceptual framework was utilised to discern PSTs' strengths and limitations in their mathematical pedagogical content knowledge with respect to the order of operations. **Figure** ***3.1*** illustrates the conceptual framework of the study.

Figure 3.1

The Conceptual Framework of the Study

The conceptual framework is presented using a multi-layered pie chart to indicate the interrelatedness among the different elements of the framework. The inner core of the chart shows the knowledge components that form mathematical pedagogical content knowledge whereas each component moving to the outer edge (the outer core) indicates the elements correspond to the knowledge components. Different colours are used in the figure for ease of reference.

Building upon previous researchers' description about knowledge of teaching (Askew et al., 1997; Ball et al., 2008; Shulman, 1987), this study defines *mathematical pedagogical content knowledge* as the blending of mathematical content and pedagogy required for teaching mathematics. Although different researchers have conceptualised the professional knowledge for teaching mathematics differently, most of them include three main elements in their conceptualisations as discussed in Section 3.2 (p. 47). The elements are knowing the specific content for teaching, understanding students' learning, and organising the content of teaching in a comprehensible manner to students. In light of this, this study conceptualises mathematical pedagogical content knowledge as an integration of three broad knowledge components namely knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches.

I acknowledge that the conceptual framework of this study is not designed in a way as broad as some other frameworks found in the literature. For example, Chick et al.'s (2006) framework is comprehensive as the framework includes many elements to describe how the pedagogical knowledge is enacted in the classroom. This is different with the present study because due to school closures as discussed in Section 1.5 (p. 7), the present study has to set boundaries and focus on the elements that can be analysed without involving classroom observations. With the three knowledge components as outlined in **Figure *3.1***, it is believed that using the conceptual framework developed in this study could add to current understanding about PSTs' professional knowledge, in particular the pedagogical content knowledge in relation to the order of operations. Accordingly, the following sections provide details of the three knowledge components.

3.4 KNOWLEDGE COMPONENTS OF MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE

To provide a clear description of the mathematical pedagogical knowledge used for teaching the order of operations, existing teacher frameworks (Askew et al., 1997;

Ball et al., 2008; Chick et al., 2006; Rowland et al., 2005; Schoenfeld, 2013; Tatto et al., 2008) are discussed by relating them to the three knowledge components outlined in the developed conceptual framework (see **Figure *3.1***).

3.4.1 Knowledge of Content

From a mathematics-specific standpoint, *knowledge of content* describes a PST's understanding of mathematical contents and procedures appropriate for teaching. This knowledge is underpinned by two elements. The first element is about explaining reasons of using or applying mathematical procedures. This element is incorporated because knowing PSTs' reasons underpinning mathematical procedures may potentially reveal their misinterpretations of the procedures. In this sense, reasoning involves the process of making logical conclusions based on evidence or assumptions about the procedures (Battista, 2017). Some existing teacher frameworks do not explicitly emphasise the reasoning of mathematical procedures (Ball et al., 2008; Rowland et al., 2005; Schoenfeld, 2013); some merely check if procedures are explained (Askew et al., 1997); and some analyse descriptions of procedures (Chick et al., 2006; Tatto et al., 2008). The conceptual framework of the study, thus, enables an investigation into the reasons used by PSTs when applying the order of operations.

The second element is about making mathematical connections between related mathematical ideas. In this respect, making connections is a way to describe how mathematical ideas are related to each other and how each idea fits into a bigger picture. Considering the potential linkages among different mathematical ideas by analysing how they originate and extend could lead to deeper understanding about mathematics (NCTM, 2000). To this end, PSTs must have the ability to identify and explain how different aspects of mathematics connect in order to support students in seeing mathematics as meaningful and sensible. This element has been highlighted in most existing teacher frameworks. For example, the KQ framework highlights the need to address coherency of mathematics by making connections between concepts or procedures (Rowland et al., 2005). The TRU framework also requires PSTs to prepare classroom discussions that provide opportunities not only for learning facts and techniques, but also for making coherent connections between mathematical ideas (Schoenfeld, 2013). Clearly, the conceptualisation of knowledge of content goes beyond general procedural fluency and takes into account the interrelatedness among mathematical ideas. In addition to examine appropriate mathematical connections as exemplified in existing teacher frameworks, the conceptual framework of the present study further analyses problematic connections PSTs make that may lead to misinterpretations or misuse of the order of operations.

Indeed, explaining reasons of mathematical procedures and making mathematical connections are equally important and dependent on each other. Both elements are perceived as relevant because they are the indicators of a PST's understanding about the mathematical contents within the mathematics curriculum as a whole (Askew et al., 1997). These elements are pertinent to knowledge of content also because they may affect how PSTs interpret the mathematical content they are expected to teach students.

3.4.2 Knowledge of Students' Sense Making

Knowledge of students' sense making refers to a PST's knowledge used for recognising the ways students make sense of mathematical procedures. In this regard, sense making involves "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (NCTM, 2009, p. 4). This definition stresses the connections between what is learned currently and what has been learned previously. It is necessary to be aware of what students' prior knowledge are and how the knowledge is developed because conceptions students bring with them might be misinterpreted and subsequently cause learning difficulties. To teach effectively, PSTs must not only focus on the mathematical convention they are teaching, but also inquire about how students build meaning about the seemingly arbitrary convention (Rowland, 2013).

This knowledge is underpinned by two elements. The first element is about analysing students' written work. Analysing students' mathematical procedures and identifying the correctness of students' written work are an essential practice embedded in teaching procedures (Baldinger, 2020). Most existing teacher frameworks include the knowledge needed to examine students' responses during classroom practices (Ball et al., 2008; Rowland et al., 2005; Schoenfeld, 2013; Tatto et al., 2008). For example, the mathematics pedagogical content knowledge framework initiated by Tatto et al. (2008) classifies the knowledge about students based on when and how the knowledge is being used in different teaching phases. They grouped two elements of similar knowledge (predicting typical students' misconceptions and diagnosing typical students' misconceptions) into two different categories: pre-active and interactive dimensions. The pre-active dimension concerns the knowledge used to structure the planning phase of teaching whereas the interactive dimension involves the implementing phase. This type of classification is beyond the scope of the present study because it is more appropriate for studies involving classroom teaching observations. No matter written work or spontaneous responses obtained in classrooms, it is important for a teacher framework to encompass the knowledge used to analyse students' responses because this practice allows PSTs to attend to strengths and weaknesses of students (Baldinger, 2020).

The second element is about interpreting students' sense making. Most existing teacher frameworks have included this element, but they do not analyse the approaches PSTs used to interpret students' sense making (Ball et al., 2008; Chick et al., 2006; Schoenfeld, 2013; Tatto et al., 2008). Knowing how PSTs interpret students' sense making is crucial because key features for supporting PSTs to understand students' learning can be identified. Using Baldinger's (2020) three approaches (mathematical, pedagogical, self-comparison) as discussed in Section 2.5.3 (p. 34), the present study, thus, examines how PSTs analyse students' written work and interpret students' sense making.

3.4.3 Knowledge of Teaching Approaches

Knowledge of teaching approaches is a PST's knowledge about organising and representing mathematical contents in ways that students may understand. This knowledge is underpinned by two elements. The first element is about planning instructional activities. Due to the importance in understanding PSTs' design of instruction, most of the existing frameworks have echoed this element both in planning to teach and in the act of teaching (Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Rowland et al., 2005; Schoenfeld, 2013; Tatto et al., 2008). The present study focuses only on the former because no teaching observation is conducted due to school closures as discussed in Section 1.5 (p. 7). It is acknowledged that much can be gained from analysing the instructional activities in planning to teach and in the act of teaching itself. However, I claimed that examining the planning is sufficient because the act of teaching is mostly based on a PST's planning to teach mathematics.

The second element is about suggesting ways to eliminate students' errors and misunderstandings. Some teacher frameworks did not specifically include pedagogical knowledge in addressing students' errors (Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Tatto et al., 2008). For example, Tatto et al. (2008) adheres to providing feedback on students' responses, but they do not include the knowledge used to resolve students' errors and misunderstandings. Although both Rowland et al. (2005) and Schoenfeld (2013) analysed PSTs' approaches in confronting and resolving students' difficulties, their frameworks primarily focus on classroom practices rather than outlining particulars ways of PSTs' teaching. Thus, this second element is included in the conceptual framework to examine PSTs' approaches in eliminate students' errors and misunderstandings in order of operations.

As belief orientations reflect a PST's preferred way of carrying out tasks and making decisions in the process of teaching, teaching approaches are described using Askew et al.'s (1997) transmission, connectionist, and discovery orientations (see Section 2.6.2, p. 41). Understanding PST's beliefs about teaching may inform pedagogical choices and strategies of teaching mathematics (Goulding et al., 2002; Rowland et al., 2005; Schoenfeld, 2007; Tatto et al., 2008). Therefore, the present study uses the three orientations to capture the depth and complexities of the mathematical pedagogical knowledge demanded for teaching the order of operations.

3.5 CHAPTER SUMMARY

Teacher knowledge frameworks in mathematics education have been reviewed to inform the conceptual framework development of this study. Although Chick et al.'s (2006) framework is comprehensive, some elements, such as explanations, are lacking specificity. Despite the relevance of using the MKT framework in analysing mathematics teacher knowledge, the MKT framework has a limitation about the definition and precision of the knowledge domains. Acknowledging there is no "one size fits all" framework due to different studies have different boundaries (Askew et al., 1997; Chin, 2013), a conceptual framework underpinning by different elements were developed to suit the purposes of the present study.

The conceptual framework of this study is formed by three main knowledge components in which each component contributes in significant ways. If a PST's knowledge of content is superficial or inaccurate, students are likely to develop limited understanding about a particular mathematical procedure. If a PST's knowledge of students' sense making is minimal, students may be expected to learn procedures that are irrelevant to their current knowledge state. If a PST's knowledge of teaching approaches is narrow, students may make errors and misinterpret mathematical procedures. Taken as a whole, these three knowledge components are considered as fundamental aspects of a PST's competence in teaching mathematics. Chapter 3 has presented the conceptual framework that guides the investigation of PSTs' mathematical pedagogical content knowledge. The next chapter considers the research design and methods of the study.

Chapter 4: Research Design

4.1 CHAPTER INTRODUCTION

The present study explores PSTs' mathematical pedagogical content knowledge of the order of operations. Sections 4.2 and 4.3 describe the theoretical perspective and research design employed to answer the following research questions:

1. *How do pre-service secondary mathematics teachers apply the order of operations to evaluate mathematical expressions?*
2. *How do pre-service secondary mathematics teachers interpret the connections between the order of operations and the properties of operations?*
3. *How do pre-service secondary mathematics teachers determine the order of operations of contextualised problems?*
4. *How do pre-service secondary mathematics teachers interpret students' written work involving the order of operations?*
5. *How would pre-service secondary mathematics teachers plan to teach the order of operations?*

Section 4.4 provides an overview of the study participants, while Section 4.5 delves into the methods implemented throughout the study, outlining the specific procedures and data collection tools utilised to gather the necessary information. The section also discusses the rationale behind the chosen methods and how they align with the research purposes. Following this, Section 4.6 covers the data analysis approach, and Section 4.7 outlines the measures taken to ensure the study's trustworthiness. Additionally, Section 4.8 presents the pilot study, and Section 4.9 engages in a discussion on the ethical considerations pertinent to the research.

4.2 THEORETICAL PERSPECTIVE

The primary purpose of this study was to gain a more comprehensive description and understanding of PSTs' mathematical pedagogical content knowledge. Specifically, the study focused on investigating how PSTs interpret the order of operations and planned the teaching of the topic. To accomplish this purpose, the study aligned itself with the interpretive paradigm, which recognises the constructive nature of knowledge and explains the situated interpretations of social reality (Crotty, 1998). Through the use of qualitative research methods, such as interviews, the study was able to reveal the complex interplay between PSTs' existing knowledge and personal beliefs that contribute to the formation of mathematical pedagogical content knowledge.

Furthermore, the present study drew theoretical inspiration from the perspective of symbolic interactionism (Crotty, 1998). This perspective emphasises the significance of interpretating symbols associated with the order of operations and highlights how PSTs engage in the construction of meaning through the interactions with students, peers, and the curriculum. Consequently, symbolic interactionism, within the interpretive paradigm, can offer specific theoretical insights into the ways in which PSTs interact, communicate, and assign meanings to symbols in the context of teaching and learning.

4.3 CASE STUDY DESIGN

A case study explores "a real-life, contemporary bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information and reports a case description and case themes" (Creswell, 2013, p. 97). The case study provides an intensive and comprehensive understanding about a bounded context (Gustafsson, 2017). In this study, the context or bounded system was Malaysian PSTs. The case study research process also allows different data of the case, such as interview transcripts, documents, and observations to be collected. More importantly, this research design may lead to transferability through giving an opportunity for other researchers to use the principles learned in a case to other cases (Schoch, 2019).

Case study research design involves a specific way of collecting, organising, and analysing data based on specific purposes to gather comprehensive, systematic, and in-depth information about the case of interest (Patton, 2002). It begins with selecting the case. As Yin (2018) stated, researchers must first define and bound the case. This can be understood as identifying the case, such as, a person, situation, or phenomenon, and determining the scope of the study. In the present study, 11 pre-service secondary mathematics teachers constitute to 11 cases. After selecting the case, the next step is determining the types of data required and the ways to collect the data.

A case study with 11 PSTs was used in this study because the research aim aligned with the principles of this research design. Given that the main aim of this study was to illuminate an understanding of PSTs' mathematical pedagogical content knowledge, rather than to generalise, a case study design was deemed appropriate. This design could facilitate the provision of a rich and in-depth description of the subtleties and complexities associated with PSTs' mathematical pedagogical content knowledge. The next section discusses the selection of the sample.

4.4 THE SAMPLE

This study was conducted at a prominent public university in Malaysia, chosen as an ideal research site due to its strong emphasis on training prospective secondary teachers for the nation. This university holds a prestigious position within the academic landscape, known for its extensive experience and accomplishments in PST education. Notably, the university offers a comprehensive range of programmes, including diploma, undergraduate, and postgraduate options, spanning multiple disciplines and faculties. Its long-standing commitment to excellence in education makes it a fitting institution for this study.

The sample of the present study was 11 pre-service secondary mathematics teachers enrolled in Bachelor of Education (Mathematics) (Hons) at this university. Mathematics was the first teaching area for the PSTs. At the time of data collection, they were enrolled in the 4^{th} year of a four-year program. **Table 4.1** presents the pseudonyms of the 11 PSTs.

Table 4.1

Case Study Participants

Case study	Pseudonym of participant
1	Audrey
2	Brendan
3	Casey
4	Danny
5	Eddy
6	Felix
7	Gavin
8	Howard
9	Irene
10	Julie
11	Kevin

The sample was selected based on a purposeful sampling method. As claimed by Patton (2002), a purposeful sampling is effective for a case study research design as it may generate an in-depth exploration about a phenomenon. Through purposeful sampling, people, situation, objects, or conditions that fulfill certain criteria are included to obtain the required information (Patton, 2002). In this study, the participants must possess mathematics courses and instruction skills in which they have taken the units that contribute to the development of mathematical pedagogical content knowledge. In this regard, they have attended mathematics courses such as Linear Algebra, Discrete Mathematics, Algebraic Structure, Calculus, and Statistics that discuss content knowledge of mathematics. In addition, they also have attended educational professional courses such as (i) Student Learning and Development, (ii) Learning Management, and (iii) Teaching, Technology and Assessment that inculcate knowledge and skills to design learning experiences and principles that underpin pedagogical practices. Being at the end of their initial teacher education also allows the participants to draw upon their knowledge gained over the period of their initial teacher education programme.

As the study is of a qualitative design, the sample "does not represent the wider population, it simply represents itself" (Cohen et al., 2007, p. 115). It is the cases that provide insights into the phenomenon of study, not the whole population (Schoch, 2019). Therefore, the data are not to generalise findings beyond the participants of this study. Rather, the data were thoroughly analysed to describe mathematical pedagogical content knowledge in the context in which it was used and learned among the participants.

4.5 METHODS OF DATA COLLECTION

Data were collected from each participant through a 40 – 60-minute individual research session via Zoom. Each research session involved a participant completing a questionnaire and a clinical task-based interview. At the beginning of the research session, the questionnaire was sent to the participant via email. At the end of the research session, the participant's written responses of the questionnaire were scanned and sent to the researcher via email. The interviews were recorded through Zoom and the recordings were subsequently transcribed for analysis.

The actual data collection process spanned a period of three months. During this time, data were drawn from three sources. The first source was a questionnaire that requires the use of the order of operations. This questionnaire comprises three open-ended items. These items are discussed in detail in Section 4.5.1 (p. 63).

The second data source consisted of audio-recordings capturing clinical task-based interviews. These interviews played a vital role in eliciting deeper insights into the participants' reasoning regarding the questionnaire items. Follow-up questions were asked to delve further into their perspectives. Furthermore, during these interviews, two specific tasks were introduced to the participants, focusing on students' common errors and examples of classroom practices. These tasks are discussed in detail in Section 4.5.2 (p. 68).

An additional data source utilised in this study was the lesson plans created by the participants. These lesson plans, developed prior to the interviews, offer further information about the participants' knowledge of teaching approaches. After the research sessions, the participants shared their lesson plans with the researcher via email. **Table 4.2** shows the data sources employed to address the five research questions in relation to the conceptual framework.

Table 4.2

Data Sources in Relation to the Research Questions and the Conceptual Framework

Mathematical Pedagogical Content Knowledge				
Knowledge of Content			Knowledge of Students' Sense Making	Knowledge of Teaching Approaches
RQ 1	RQ2	RQ3	RQ4	RQ5
Item 1	Item 3	Item 2	Task 1	Task 1 (iii), Task 2, Lesson plans

Note. RQ represents research question.

4.5.1 Questionnaire

The questionnaire, provided in Appendix A (p. 217), begins by gathering background information such as academic qualifications, specialist teaching areas, and grade point average. Subsequently, the questionnaire incorporates three open-ended items aimed at probing the participants' knowledge of content about the order of operations. The utilisation of open-ended items was intentional, as it helps minimise the potential impact of chance factors and ensures a more accurate representation of the results. This approach enables a comprehensive exploration of participants' knowledge and allows for a richer analysis of their responses.

The first item was designed to determine how PSTs use their knowledge of the order of operations to evaluate mathematical expressions. Specifically, this item was developed to answer Research Question 1 in relation to knowledge of content. This item consisted of eight mathematical expressions (Expressions 1a – 1h) as presented in **Table 4.3**.

Table 4.3

Item 1 of the Questionnaire

1)	Evaluate the following expressions without using a calculator. Write down precisely and clearly every step to reach your answer.
a)	$10 \div 5 \times 2 =$
b)	$4 \times 6 \div 3 =$
c)	$3 - 12 + 8 =$
d)	$4 + 17 - 7 =$
e)	$10 - 3^2 =$
f)	$2^{3^2} =$
g)	$8 - \left((-2) + 4\right) =$
h)	$(9 - 2)(7 - 4) =$

This item was designed as representative of most situations involving order of operations, such as multiplication and division ($4 \times 6 \div 3$). However, no expression containing several operations (e.g., $1 + 2 \times 3 - 4 \div 5$) was included because more items might exhaust participants' concentration. The inclusion of eight expressions was deemed sufficient for understanding how participants evaluate mathematical expressions and to gain insights into their underlying reasoning behind the order of computations. Considering a balance between the complexity of the mathematical expressions and the cognitive load placed on the participants, this item was designed to effectively capture the PSTs' knowledge of content replated to the order of operations.

The participants were required to simplify the expressions by showing all of their work, including the intermediate steps required for reaching a solution to each of the expressions. Items 1a – 1d were designed to analyse PSTs' knowledge about the left-to-right order. For expressions involving addition and subtraction (or multiplication and division), the correct order of operations is from left to right. However, for expressions of the form $a + b - c$ and $a \times b \div c$, the same answer can be obtained no matter if the order of computations is from left to right or from right to left. These four expressions were included to understand how PSTs applied the order of operations, particularly to simplify expressions that could be evaluated from any orders.

Items 1e and 1f were formulated to analyse PSTs' knowledge about operating on indices. Drawing inspiration from Pappanastos et al.'s (2002) study, specifically their investigation of participants' comprehension of the order of operations using the expression $8 - 5^2$, the present study adapted and designed Item 1e as $10 - 3^2$. By utilising this expression, the study seeks to uncover the underlying reasoning that participants employ when evaluating expressions involving indices.

Items 1g and 1h were used to examine PSTs' knowledge about the use of parentheses when executing the order of operations. Parentheses can be used to represent multiplication, for example, $(-2)(-4)$ implies that -2 multiply by -4. Item 1h was included to investigate how participants used their knowledge of parentheses in evaluating expressions of the form $(a)(b)$. All the eight expressions could provide information about the PSTs' knowledge of the order convention, starting from subtraction to parentheses. Based on the responses provided both in the questionnaire and in the interviews, the reasons of using the order of operations were identified.

Item 2 of the questionnaire focuses on examining how PSTs apply the order of operations when mathematising contextualised problems. This particular item was designed to address Research Question 3, which pertains to the participants' knowledge of content. It consisted of four contextualized problems (Problems 2a – 2d), which are detailed in **Table 4.4**. For each of these contextualised problems, the PSTs were required to write a single mathematical expression that accurately represents the given context.

Table 4.4

Item 2 of the Questionnaire

2)	Write a mathematical expression to represent each of the following problems.
a)	A tyre factory produces 6351 tyres every 3 days. How many tyres will the factory produce in 14 days?
b)	There are 532 parking spots on the first level of a multi-level parking lot and the rest of the parking spots are distributed equally on the other 8 levels. How many parking spots are there on the top level if there are total of 1532 parking spots?
c)	In a bookshop, paperback books cost \$3 and hardback books cost \$4 each. Alice buys six paperback and two hardback books. How much change will Alice receive from a \$50 banknote?
d)	Helen is enlarging a photo of 4cm width on her tablet screen. The width of the photo is doubled each time she enlarges the photo. What is the width of the photo on her screen if she enlarges the photo five times?

Each item involved different mathematical operations. Item 2a involved multiplication and division; Item 2b involved subtraction and division; Item 2c involved addition, subtraction, and multiplication; Item 2d involved operating on indices and multiplication. Jeon (2012) provided a mathematical expression to their teachers and required them to write a problem based on the expression. In contrast to Jeon's (2012) study, Item 2 of the questionnaire requires participants to write expressions to best reflect the given contextualised problems. This item is considered important because the need to mathematise problems into expressions has been acknowledged in many national curricula such as ACARA (2018).

As there are different categories of contextualised problems as discussed in Section 2.4.2 (p. 28), Items 2a and 2b are based on the characteristics of essential and relevant contextualised problems whereas Items 2c and 2d are based on the characteristics of camouflage contextualised problems. Based on the responses provided both in the questionnaire and in the interviews, the ways PSTs used to determine the order of operations of contextualised problems could be examined.

Item 3 of the questionnaire was designed to examine PSTs' knowledge about the connections between order of operations and properties of operations. This item was developed to answer Research Question 2, which pertains to their knowledge of content. This item consisted of two hypothetical situations (Items 3a and 3b) as presented in **Table 4.5**.

Table 4.5

Item 3 of the Questionnaire

3)	Answer the following questions.	
a)	Lisa and Richard evaluate the expression $\mathbf{4+3-2}$ differently. They ask you whose solution is correct.	
	Lisa's response $4+3-2=7-2$ $=5$	Richard's response $4+3-2=4+1$ $=5$
	(i) Assess the students' responses.	
	(ii) Explain why Lisa and Richard can obtain the same answer even though they evaluate the expression in different ways.	
b)	Olivia and Desmond evaluate the expression $4(2+3)$ differently. They ask you whose solution is correct.	
	Olivia's response $4(2+3)=4(5)$ $=20$	Desmond's response $4(2+3)=4(2)+4(3)$ $=8+12$ $=20$
	(i) Assess the students' responses.	
	(ii) Explain why Olivia and Desmond can obtain the same answer even though they evaluate the expression in different ways.	

Item 3a examines the connections between the left-to-right order and the associative and inverse properties. Item 3b analyses the connections between the order of operations and the distributive property. Through Item 3, the ways the PSTs used to interpret the connections between the order of operations and the properties of operations could be determined.

4.5.2 Clinical Task-based Interview

A clinical task-based interview involves an interviewer giving an interviewee open-ended task to complete while the interviewee is thinking aloud. The interviewer then asks further questions according to the interviewee's responses. This is the strength of a clinical task-based interview, the possibility to post further questions when the interviewee raises an issue of concern. Moreover, a clinical task-based interview is appropriate to gain more in-depth data on participants' understandings by providing an opportunity to share and define their perspectives and experience (Zazkis & Hazzan, 1998). It also enables "the discovery of cognitive processes, the identification of what is behind these cognitive processes, and the evaluation of the interviewee's competence" (Ginsburg, 1981, p. 5).

Clinical task-based interviews have been used in a number of studies to understand teacher knowledge. The study of Jenkins (2010), for example, employed an open-ended mathematical task during the interviews to examine PSTs' knowledge about students. Jenkins noted that a clinical task-based interview was fittingly created to observe participants' reasoning directly. Another study conducted by Charalambous and Hill (2012) required teachers to solve Mathematical Knowledge for Teaching items and explain their thinking during clinical task-based interviews. Through the interviews, they successfully gleaned in-depth information about the knowledge needed for teaching mathematics. With the intent to involve PSTs in the articulation of their tacit knowledge, a clinical task-based interview approach is deemed appropriate for data generation of the present study.

In this study, the clinical task-based interviews started by asking follow-up questions to gain further insights into participants' reasoning about the questionnaire items. In the interviews, the participants could clarify their responses and provide more detailed explanation. The interviews were driven by an interview schedule (see

Appendix C, p. 224). The role of the interviewer was to follow up the participants' responses by seeking clarification and provide scaffolding when necessary. Then, two tasks involved students' common errors and examples of classroom practices in relation to the order of operations were presented to the participants. These tasks (see Appendix B, p. 221) are discussed in the following paragraphs.

To explore how the PSTs interpret students' written work involving the order of operations, two hypothetical incorrect student responses were presented as Task 1 (see **Table 4.6**). Specifically, Task 1 was developed to answer Research Question 4 in relation to knowledge of students' sense making.

Table 4.6

Task 1

Michelle is a student in Grade 7. The following are her responses to two expressions.
a) $$18 \div 3 \times 2 = \frac{18}{3 \times 2} = \frac{18}{6} = 3$$
b) $$\frac{50 + 28}{5 + 7} = \frac{10 + 4}{1 + 1} = \frac{14}{2} = 7$$
i) Assess Michelle's responses. ii) Why do you think she evaluated the expression that way? What misunderstandings is she likely to have? iii) How would you teach to avoid students from making errors and misinterpreting the order of operations?

The first student work (Task 1a) reflected the error of performing multiplication before division. This error was due to the student interpreting $18 \div 3 \times 2$ as $18 \div (3 \times 2)$ and had given priority to multiplication. The second student work (Task 1b) presented the error of calculating division before addition for an expression that was in a fraction form. The student cancelled common factors even though the numerator and denominator were both sums.

Participants were required to determine the accuracy of the work, deduce students' thinking, and offer instructional strategies. It is noted that the question "How would you teach to avoid students from making errors and misinterpreting the order of operations?" was used to explore how PSTs suggest ways to eliminate students' errors and misunderstandings of the order of operations. By focusing on this aspect, the study aimed to gain valuable insights into the PSTs' ability to identify potential pitfalls and offer effective instructional approaches to enhance students' understanding and application of the order of operations.

In order to ascertain the preferred approach that PSTs believe is most suitable for teaching the order of operations, three distinct classroom practices were presented as Task 2. The task is provided in **Table 4.7**. Participants were required to choose and justify the classroom practice they deemed most effective. Each classroom practice represents a distinct teaching approach, namely transmission, connectionist, and discovery.

To facilitate ease of reference during the interview and data analysis, fictional characters named Tracy, Connie, and Dickson were assigned to represent the transmission, connectionist, and discovery approaches, respectively. Participants were required to give opinions on the most effective practice then choose and justify which classroom practice most likely they will use to teach the order of operations. This task allows for an exploration of participants' preferences and reasoning behind their instructional decisions, shedding light on their pedagogical perspectives. Together with the lesson plans developed by the PSTs, the data collected from the Task 1(iii) and Task 2 in the interviews could provide sufficient information about PSTs' knowledge of teaching approaches. Specifically, these tasks and lesson plans may answer Research Question 5.

Table 4.7

Task 2

<table>
<tr><td colspan="2">Three teachers, Tracy, Connie, and Dickson each conducts an introductory lesson on the order of operations in three different 7th Grade classrooms. Their classroom practices are shown as below.</td></tr>
<tr><td>Teachers</td><td>Classroom practices</td></tr>
<tr><td>Tracy</td><td>Tracy writes the mnemonic, BIDMAS, on the board to introduce the order of operations. She explains that priority must be given to brackets before indices, followed by division and multiplication in order of appearance from left to right, then addition and subtraction in order of appearance from left to right. Students will need to memorise this mnemonic. She writes the expression $24 \div 6 \times 2$ on board and says "So, BIDMAS tells us that operations are to be carried out from left to right when we have division and multiplication in the expression." She continues, "I have to divide 24 by 6 first (pointing at $24 \div 6$) to get 4 (write 4×2), then 4 times 2 so I get 8 (write 8).

$$24 \div 6 \times 2 = 4 \times 2 = 8$$

Students are then given a number of expressions and told to evaluate based on BIDMAS. As Tracy moves around the class, she gives more expressions for students to evaluate.</td></tr>
<tr><td>Connie</td><td>In a lesson, Connie has set up 7 stations in which each station contains 3 cubes and 5 spinners. Connie requires students to calculate the total number of cubes, total number of spinners, total number of items in each station, and total number of items used for the whole lesson. Then she asks the students to form an expression that can be used to calculate the total number of items used for the whole lesson. Students work in pairs using a variety of methods. As they begin to complete the task, Connie brings the class together and invites students to provide the answers and explain the method used. The other students are attentive to these</td></tr>
</table>

	explanations. Students' errors are discussed so that a more efficient method can be identified. Connie brings in the idea of brackets in helping students to refine their methods. They continue to discuss problems that involve different operations. Students develop an understanding of the order of operations through working on given problems and whole class discussion.
Dickson	Dickson organises students in groups. He gives a number of expressions to all the students and requires them to evaluate the expressions by any method. The expressions are as follows: $8 + 7 - 4 =$ $16 - 9 + 5 =$ $12 \times 6 \div 3 =$ $45 \div 5 \times 9 =$ $4 + 3 \times 10 =$ $(25 - 3) \div 11 =$ $2 \times (24 \div 6) =$ $4 \times 3^2 =$ $(-2)^3 - 10 =$ Answers to all the expressions are provided. In groups, students evaluate the expressions but some of them obtain different answers. They are surprised that some expressions have several different answers. They compare their answers with the answers provided by Dickson then spend some time to recalculate and discuss within their groups. Based on evaluating the expressions, the students notice a pattern that leads them to generalise the correct order to perform computations. a In your opinion, which classroom practice is the most effective practice to teach the topic? Why? b Which classroom practice do you prefer to use? Why?

In preparing the interview questions, careful consideration must be taken for setting questions that are not being prejudicial and that are not considered as ambiguous, leading, double-barrelled, assumptive, hypothetical, or overly personal (Cohen et al., 2007). In other words, the interview questions must free from any biases or ambiguities. As a clinical task-based interview requires the interviewer to post contingent follow-up questions based on the interviewee's spontaneous responses, there is a possibility of unintentionally asking leading questions influenced by preconceived notions (Inglis, 2006). Swanson et al. (1981), however, explained that if careful measures are taken accordingly, the potential issues affecting clinical task-based interviews are not substantial compared to other methods of data collection. Hence, the present study can gather reliable data through carefully conducting clinical task-based interviews following the interview schedule.

4.5.3 Lesson Plan

Lesson planning is essential for understanding knowledge of teaching approaches because how PSTs plan the lesson reflects how they teach the lesson (Buchbinder & McCrone, 2020). Realising the importance of lesson planning, the lesson plans of participants were collected. These lesson plans, developed prior to the interviews, specifically focused on the teaching of the order of operations. In particular, the lesson was planned for students of Grade 7. By analysing these lesson plans, valuable insights can be obtained regarding PSTs' pedagogical strategies, instructional techniques, and the extent to which they incorporate effective teaching approaches in their lesson design. The learning standards associated with the lesson plans are detailed in **Table 4.8**.

Table 4.8

Learning Standards of the Order of Operations (MOE, 2016)

Content standard	Learning standards
1.2 Basic arithmetic operations involving integers	1.2.3 Perform computations involving combined basic arithmetic operations of integers by following the order of operations.
	1.2.4 Describe the laws of arithmetic operations which are Identity Law, Communicative Law, Associative Law and Distributive Law.
	1.2.6 Solve problems involving integers based on the order of operations

4.6 DATA ANALYSIS

The process of thematic analysis has been encapsulated by researchers differently (e.g., Ary et al., 2013; Birks & Mills, 2015; Braun & Clark, 2013; Smith et al., 2009; Willig, 2013). For example, Ary et al. (2013) identifies the process of data analysis as (1) organising data, (2) coding, reducing, and generating theme, and (3) interpreting and presenting data. The study of Braun and Clarke (2021) proposes reflexive thematic analysis which is theoretically flexible in which it can be used to inform different frameworks and different research questions. The reflexive thematic analysis is a six-phase process to facilitate the development of patterns across several cases: (1) data familiarisation, (2) data coding, (3) initial theme generalisation, (4) theme development and review, (5) theme refining, defining, and naming, and (6) writing up. Data of the present study were analysed following the steps suggested by Ary et al. (2013) and using the approaches proposed by Braun and Clarke (2021).

4.6.1 Organising Data

The first phase is organising data. This phase entails reading and re-reading the entire dataset in order to help researchers to make sense of the data in relation to their research questions. In this study, I first organised the data based on data types (texts from questionnaire and lesson plans, audio files). I read and re-read the texts from the questionnaire and lesson plans to make sense of them.

I familiarised myself with the interview data by listening to each interview recording before transcribing the recording. I listened to the first playback of each interview recording to develop an understanding of the main areas of discussion. Immediately after listening to the first playback, I transcribed each audio recording to text using Microsoft 365. By referencing to the transcripts generated from Microsoft 365, I manually transcribed the audio recording again so that the responses given in Malay were translated into English. To ensure accurate data translation, I used Weeks et al.'s (2007) translation process: determine the context, forward translation, backward translation, examine the meanings, and revisit the translation process to get similar interpretations. **Table 4.9** shows a translation sample.

Table 4.9

Translation Sample

Question	Responses in Malay	Transcribed into English
What makes you think that Desmond is wrong and Olivia is correct?	Desmond betul kalau dia letakkan unknown di dalam, jadi dalam soalan ini, Desmond salah, Olivia betul. Kita biasa buat seperti Desmond kalau terdapat satu unknown contoh x atau y dalam kurungan. Contohnya, maybe $4(x+3)$. Jadi, Desmond salah kerana cara dia adalah untuk soalan yang mempunyai unknown. Olivia betul kerana dia mengikut urutan untuk menyelesaikan terlebih dahulu apa yang terdapat dalam kurangan.	Desmond is correct if he puts an unknown inside, so in this question, Desmond is wrong, Olivia is correct. We used to do like Desmond if there is an unknown like x or y inside the bracket. For example, maybe $4(x+3)$. So, Desmond is wrong because his method is for questions with unknown. Olivia is correct because she follows the order to solve what is in the bracket first.

The audio recordings were transcribed orthographically, noting infections, breaks, and pauses (Braun & Clarke, 2013). After complete transcribing all the audio recordings, I read and re-read the transcripts. Preliminary notes were made as a record of initial trends observed in the transcripts.

4.6.2 Coding, Reducing, and Generating Theme

The second phase is coding, reducing, and generating theme. This phase first involves generating codes that represent important feature of the data that may be of relevance to the research questions. The study of Braun and Clarke (2006) suggests two approaches of coding: “data-driven” and “theory-driven” (p. 18). The former involves researchers generate codes from data (inductive) whereas the latter involves researchers code according to literature (deductive).

In this study, I used both coding approaches to avoid missing important data. For example, based on Baldinger’s (2020) approaches to interpret students’ written work, the themes (mathematical, pedagogical, self-comparison) were adapted and refined to identify PSTs’ knowledge of students’ sense making. Data were then coded based on these existing themes through deductive theory-driven thematical analysis. On the other hand, **Table 4.10** shows an example of the preliminary coding process for inductive data-driven thematic analysis in which codes were generated from data. As presented in the extract of coding, the texts were highlighted and assigned to each code. For example, the text “gave a lot of explanations to introduce the rules” was assigned to code [c1] that is introductory explanation is an important aspect of teaching. The preliminary codes were subsequently redefined.

Table 4.10

Extract of Coding

Interview transcription	Code
Tracy is good because she gave a lot of explanations to introduce the rules [c1].	[c1] Introductory explanation is an important aspect of teaching.
She used BIDMAS is a very good way to remember the rules [c2].	[c2] Rote memorisation is essential for teaching procedures.
Moreover, a lot of questions were given to students to try so that they can see the pattern of calculations clearly [c3].	[c3] Drilling to perform procedures is common in teaching procedures.
By using BIDMAS, they can solve every question step by step easily [c4].	[c4] BIDMAS is used as a tool to perform procedures.

Text from lesson plan	Code
Allow students to take five and drink water [c5].	[c5] Giving breaks to students.
The teacher discusses with the whole class what are the possible ways to evaluate expressions involving addition and subtraction [c6].	[c6] Negotiating methods between the teacher and students.
Then, in pairs, students discuss another question involving multiplication and division [c7].	[c7] Negotiating methods between students.

After repeated iterations of coding, codes that were conducive were used to interpret themes, codes that were not informative in addressing the research questions were discarded, and codes that shared a similar meaning were combined. For example, the code [c5] was removed whereas codes [c6] and [c7] were combined. When the same code prevalent throughout the dataset, the code was considered informative and useful in developing a theme. This helps to identify "aggregated meaning and meaningfulness across the dataset" (Byrne, 2022). Emerging sub-themes were grouped to form broader themes. **Table 4.11** shows an example of generating themes.

Table 4.11

Examples of Generating Theme

Sub-theme	Theme
Transmission/traditional/teacher-centred Connectionist/student-centred	Teaching approaches of the order of operations
Division is multiplicative inverse Fraction bar as grouping symbol	Mathematical interpretation

4.6.3 Interpreting and Presenting Data

The third phase is interpreting and presenting data. In this phase, the generated themes were used to inform an overall understanding of the cases. Guided by the conceptual framework presented in Chapter 3 (p. 47), data were interpreted to show "the understanding and insight you derive from a more holistic, intuitive grasp of the data and the insights they reveal" (Simons, 2009, p. 117).

In this study, data from all data sources were compared based on themes identified in the second phase. For example, within the theme "teaching approaches of the order of operations", interview data and data collected from lesson plans were compared. Comparing data within each case and between cases provides information on the interrelationship of themes. The process of data interpretation identified mathematical pedagogical content knowledge needed to approach the order of operations among the 11 cases.

4.7 TRUSTWORTHINESS

In qualitative research, the concepts of validity and reliability are perceived differently and are conceptualised as trustworthiness, rigor, and quality (Golafshani, 2003). Instead of using the terms validity and reliability, Lincoln and Guba (1985) identified credibility, transferability, dependability, and confirmability as indicators of trustworthiness, which are deemed more suitable for an interpretive method of qualitative research. These aspects are discussed in turn.

Credibility refers to whether or not the findings of a research are credible from the perspectives of researchers, participants, and readers (Bloomberg & Volpe, 2008). In this sense, triangulation is one method to establish credibility in which researchers search for convergence among multiple data sources (Creswell & Miller, 2000). The credibility of this study was supported through different data collection tools including a questionnaire, clinical task-based interviews, and lesson plans. In addition, the credibility was strengthened by the use of member checking and peer debriefing. The participants were provided the opportunity to review their interview transcripts and frequent peer debriefing sessions with thesis supervisors allowed the researcher to develop suggested ideas. The credibility is ensured in which the supervisors scrutinised reporting and discussions of emerging themes, and consistently questioned the plausibility of the analysis and subsequent findings (Merriam, 2002).

Transferability refers to the degree to which the same phenomenon can be transferable to another contexts or settings (Bloomberg & Volpe, 2008). The transferability of this study was supported through providing thick and in-depth descriptions of the study setting, the study context, and the participant selection. Moreover, an interview procedure was prepared in detail to enable readers to transfer this study to their intended settings.

Dependability is about the stability of the research results over time and is closely corresponded to the concept of reliability in quantitative research. The dependability of this study was supported through examining the analysis process, checking the accuracy of records, and making sure the analytical techniques were used accordingly. As argued by Lincoln and Guba (1985), a demonstration of credibility is usually sufficient to establish dependability. In this regard, triangulation between multiple data sources that was discussed in relation to credibility also increased the dependability of this study.

Confirmability entails ensuring findings are derived from the experiences and ideas of participants (Shenton, 2004) and the results are not altered by researcher bias (Lincoln & Guba, 1985). In this regard, an audit trail may promote confirmability (Yin, 2009). The audit trail of this study were audio recordings of interviews and the interview transcripts to enable objective substantiation of the data. To achieve openness in terms of analytical transparency, this study provided a detailed description of how initial coding were arrived and how the data were drawn to conclusions. In a broader sense, triangulation to reduce the researcher bias, member checking to ensure the accuracy of the collected data, peer debriefing to reduce bias during analysis, and in-depth methodological description provided the evidence in supporting the confirmability of this study.

4.8 PILOT STUDY

A pilot study was conducted to refine the data collection instruments and data collection process before conducting the main study (Creswell, 2012). In the present study, the questionnaire and task-based interviews were piloted. I evaluated the pilot study in terms of the feasibility of the two data collection instruments and the data collection process. Four PSTs from the public university participated in the pilot study.

Through the piloting, any confusing wording and expression identified in the questionnaire and task-based interviews were refined. For example, considering Item 3a(i) might induce the participants to think that only one response was correct, the item was changed from “Who is correct?” to “Assess the students’ responses.” A similar refinement was also used for Item 3a(ii). Another example is about the word “parentheses”. It is noted that some participants were not familiar with this word. Instead of using the word parentheses, I will use the word “brackets” during the main data collection.

The pilot study helped me to understand the process of data collection using an electronic means. Interviewing participants via Zoom was new to me. With the pilot study, it made me familiar with how to record data effectively through Zoom.

Based on the pilot study, two main issues regarding the data collection process were identified. First issue was the internet connection. Three of the participants were disconnected from Zoom during data collection. They managed to reconnect within one or two minutes. This issue did not affect the recordings because the recordings continued automatically after the participants reconnected. However, to avoid participants from having this issue in the main study, I will ask participants whether they have a stable internet connection and remind them to check their internet connections prior starting a research session. If the internet connection problem persists and the participants have not started answering the questionnaire, I will arrange another date for data collection with them. However, if the internet connection problem persists and they have started answering the questionnaire, the research session will be terminated. No further session will be arranged for these participants. The incomplete data collected will not be analysed.

Second issue is about English competency. The participants' English competency has limited them from expressing their ideas. They used Malay occasionally when they could not express their thoughts in English. I did not insist the use of English because their ideas and thinking were more important. I will allow PSTs to use Malay in the main study then I will translate the collected interview data to English.

4.8.1 Analysis of the Pilot Data

The analysis of the pilot study shows that the developed conceptual framework could facilitate an understanding of PSTs' mathematical pedagogical content knowledge. The following paragraphs present a brief analysis of some pilot data. The analysis of some items is reported in this section because these items show qualitatively different responses. Pseudonyms P1, P2, P3, and P4 were used to represent the four PSTs participants in this pilot study.

4.8.1.1 Item 1 of the Questionnaire

Expressions 1a to 1d were designed to analyse PSTs' knowledge about the left-to-right order. The expressions were posed as follows:

$1a.\quad 10 \div 5 \times 2$

$1b.\quad 4 \times 6 \div 3$

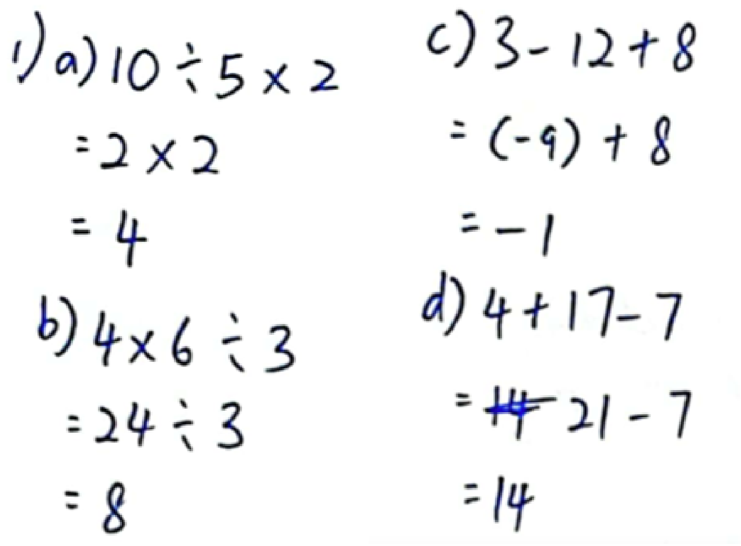
$1c.\quad 3 - 12 + 8$

$1d.\quad 4 + 17 - 7$

The correct order to evaluate these expressions is calculating from left to right. P3 and P4 used the correct order, but P1 and P2 used an incorrect order. P3's responses, reproduced in **Figure *4.1***, illustrate the ways that P3 and P4 evaluated the expressions from left to right.

Figure 4.1

P3's Responses for Expressions 1a to 1d

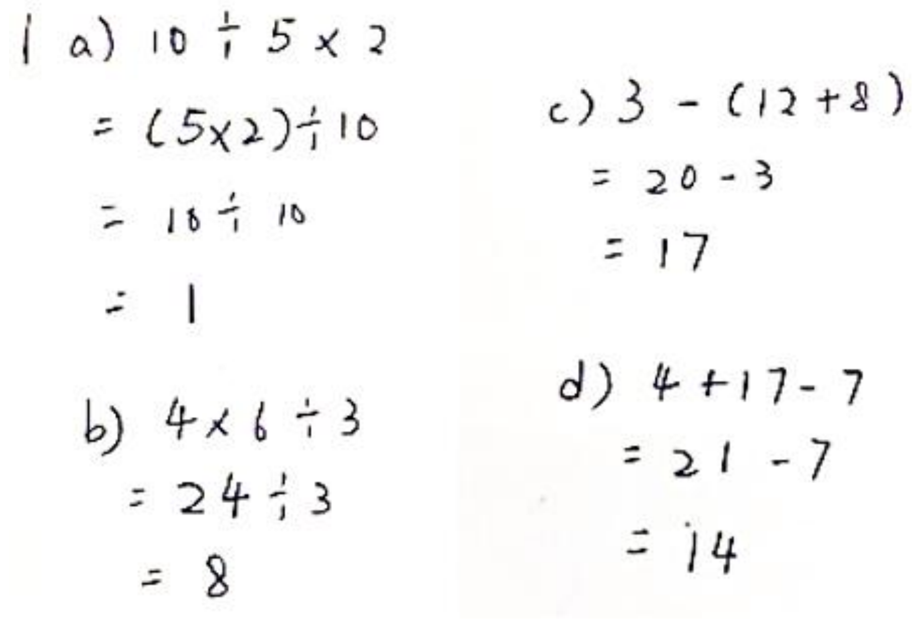


When required to give a reason for his ways of evaluation, he referred to BODMAS by stating, "We must follow BODMAS. BODMAS explains that expressions with multiplication and division, or addition and subtraction, must be calculated from left to right." This excerpt shows that P3 interpreted the left-to-right order based on his understanding of the acronym BODMAS.

In evaluating these expressions, P1 and P2 used an incorrect order. For example, P2 gave priority to multiplication over division for Expressions 1a and 1b. He also gave priority to addition over subtraction for Expressions 1c and 1d. P2's responses, reproduced in **Figure *4.2***, illustrate the ways that PSTs used an incorrect order of operations.

Figure 4.2

P2's Responses for Expressions 1a to 1d



P2 explained that, "In school many years ago when I learned the order, I was told to do multiplication before division and do addition before subtraction." This excerpt suggests that P2 misinterpreted the order of operations. His reason for giving the priority to multiplication or addition was relied on his prior experience when learning the order of operations.

Expressions $1e$ and $1f$ were formulated to analyse PSTs' knowledge about operating on indices when executing the order of operations. The expressions were posed as follows:

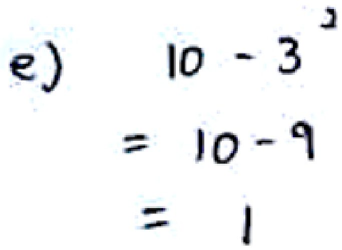
$1e.\quad 10 - 3^2$

$1f.\quad 2^{3^2}$

All PSTs used the correct order of operations to evaluate Expression 1e. P1's response, reproduced in **Figure *4.3***, illustrates the way that all the PSTs used the correct order to evaluate the expression.

Figure 4.3

P1's Response for Expression 1e

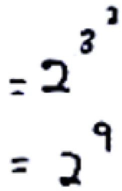


P1 explained that, "Power 2 is the index of 3. Before finding the difference, we must find 3 squared first because it is 3 times 3." This excerpt suggests that P1 recognised indices mean repeated multiplication. She operated the index before subtraction based on her understanding about indices.

The correct order of operations for Expression 1f is calculating from right to left. Only P1 used the correct order to evaluate this expression. **Figure *4.4*** shows P1's response for Expression 1f.

Figure 4.4

P1's Response for Expression 1f

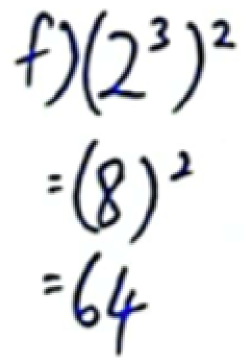


P1 explained that, "BODMAS indicates that power needs to be calculated first. Since this question involves power of power, so we must do power of power first." This excerpt implies that she interpreted the order of operations for stacked exponents based on BODMAS and her understanding about operating indices.

On the other hand, P2, P3, and P4 evaluated Expression 1f from left to right, which is an incorrect order of operations. P3's response, reproduced in **Figure *4.5***, illustrates the way that PSTs used an incorrect order of operations for this expression.

Figure 4.5

P3's Response for Expression 1f



He explained that, "We need to calculate this kind of expression from left to right, just like multiplication and division." This excerpt indicates that P3 misinterpreted the order of operations for stacked exponents as the same with that for expressions involving multiplication and division. The left-to-right order was problematic for P3 in this case.

Expressions 1g and 1h were designed to examine PSTs' knowledge about the use of parentheses when executing the order of operations. The expressions were posed as follows:

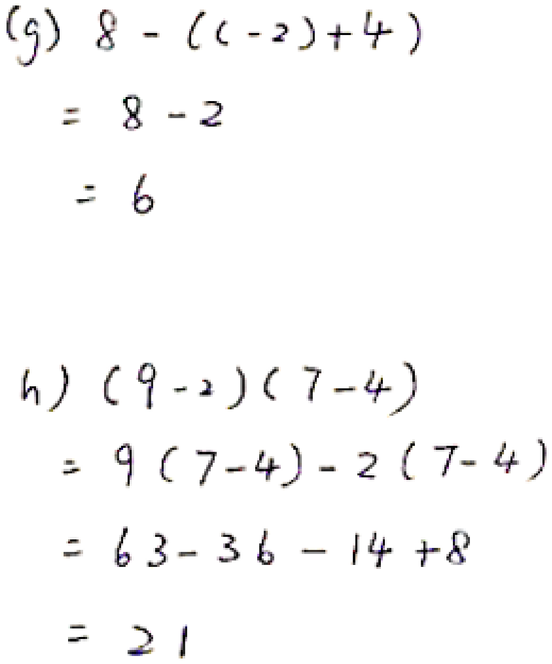
$1g. \quad 8-\left((-2)+4\right)$

$1h. \quad (9-2)(7-4)$

The correct order to evaluate these expressions is executing the operations inside the parentheses before performing other operations. All the PSTs used the correct order of operations. P2's responses were reproduced in **Figure *4.6***.

Figure 4.6

P2's Responses for Expressions 1g and 1h



P2 explained that, "Operations inside a bracket must be calculated first. Take 1h for example, we can used factorisation to do this question. The answer will be the same if we do the operation inside in the brackets first." Using factorisation to simplify this expression requires the understanding of distributive property. This excerpt implies that P2 made sense of the order of operations by connecting it to the distributive property.

4.8.1.2 Item 2 of the Questionnaire

In this section, Item 2d was analysed to show how PSTs determined the order of operations when mathematising contextualised problems. The fourth contextualised problem (Problem 2d) involved operating on indices and multiplication and was posed as follows:

Helen is enlarging a photo of 4 *cm width on her tablet screen. The width of the photo is doubled each time she enlarges the photo. What is the width of the photo on her screen if she enlarges the photo five times?*

Problem 2d can be represented using the expression 4×2^5. P2 and P4 gave a correct expression but P1 and P3 provided an incorrect expression. P2's response, reproduced in **Figure *4.7***, illustrates the way that PSTs used a correct expression.

Figure 4.7

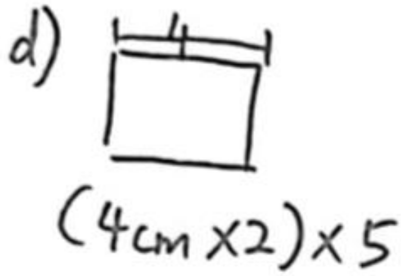
P2's Response for Contextualised Problem 2d

P2's based his explanations on the context of the problem. He explained that, "To enlarge 5 times, I used times 2 to the power of 5 because every time I enlarge the width will doubled." He correctly defined double as multiplying 2 and recognised the action of enlarging 5 times as $\times 2^5$. He managed to write a correct expression for contextualised Problem 2d based on the understanding of the underlying structures of the problem and the understanding about indices.

On the other hand, P1 and P3 gave an incorrect expression to reflect Problem 2d. For example, P3 wrote $(4 \times 2) \times 5$ for this problem (see **Figure *4.8***). P3 wrote $(4 \times 2) \times 5$ and referred "the width is doubled" as 4 times 2. He used brackets to emphasise the product of 4×2. It is likely that P3 did not completely understand Problem 2d and struggled with comprehension of the context of this problem.

Figure 4.8

P3's Response for Contextualised Problem 2d

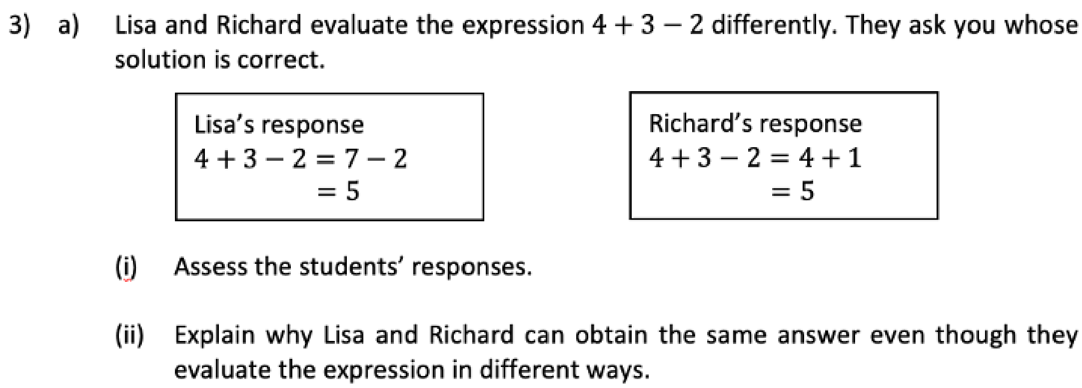


4.8.1.3 Item 3 of the Questionnaire

In this section, Item 3a was analysed to show how PSTs related the left-to-right order to the associative property of addition and the additive inverse. The item is presented in **Figure *4.9***.

Figure 4.9

Item 3a of the Questionnaire



3) a) Lisa and Richard evaluate the expression $4 + 3 - 2$ differently. They ask you whose solution is correct.

Lisa's response	Richard's response
$4 + 3 - 2 = 7 - 2$ $= 5$	$4 + 3 - 2 = 4 + 1$ $= 5$

(i) Assess the students' responses.

(ii) Explain why Lisa and Richard can obtain the same answer even though they evaluate the expression in different ways.

Of the four PSTs, only P4 used the additive inverse and associative property of addition to explain the order of operations. He stated that, "Both the students are correct because we can do this question using any orders if you change the subtraction to addition, that is, change minus 2 become plus -2." This excerpt shows that P4 recognised subtraction as additive inverse. Based on his knowledge about associativity, he recognised that the expression can be evaluated using any orders.

On the other hand, P1 explained that both solutions were correct because of the same answer. He stated that, “Lisa calculated addition first, but Richard did subtraction first. Although their orders of operations are different, they obtained the same answer 5. This means both of them are correct.” This excerpt indicates that P1 emphasised on the end result of the computation. He appeared to not recognise why both orders work.

4.8.1.4 Task 1 of the Task-based Interview

In this section, Task 1 of the task-based interview was analysed to show how PSTs interpreted students’ written work involving the order of operations. The first hypothetical student work reflected the error of performing multiplication before division (see **Figure *4.10***). This error was due to the student interpreting $18 \div 3 \times 2$ as $18 \div (3 \times 2)$ and had given priority to multiplication.

Figure 4.10

First Hypothetical Student Work of Task 1

$$18 \div 3 \times 2 = \frac{18}{3 \times 2}$$
$$= \frac{18}{6}$$
$$= 3$$

i) Assess Michelle’s responses.

ii) Why do you think she evaluated the expression that way?
What misunderstandings/misconceptions is she likely to have?

P3 and P4 interpreted the student’s written work pedagogically. Interpreting written work through pedagogical approach means drawing on knowledge about what students might commonly do and how a mathematical concept might generally be taught. P4, for example, explained that, “I think this is incorrect. The student made a common mistake and misinterpreted multiplication precedes division. This question should be calculated from left to right.” This excerpt suggests that P1 drew on his knowledge about students’ common error in the order of operations when analysing the written work.

P1 interpreted the student’s written work based on self-comparison approach. Interpreting written work through self-comparison means searching for similarities and differences between PSTs’ own solution and those of students. P1 explained that, “The student does what I did. We did 3 times 2 before doing division.” This excerpt indicates that P1 compared the student’s solution to his own solution. However, his perception that multiplication must be calculated before division was incorrect. He made the same error as the student.

4.8.2 Summary of Pilot Analysis

The analysis of the pilot study showed that the participants’ responses fit within the conceptual framework. The research instruments proved suitable to gather in-depth data related to mathematical pedagogical content knowledge of the order of operations. In other words, the instruments appeared to elicit suitable data in order to answer the formulated research questions.

The analysis, however, posed an interpretive dilemma. Take the following response for example, “She calculated 3 times 2 so is 6 and 18 divided by 6 is 3. In fact, her calculation is the same as mine in Question 1a so I can see that her working is correct.” The first part of this excerpt may imply that the participant used a mathematical approach, as he followed the student’s path of solution to check if the path made sense. The second part of the excerpt may indicate that the participant used a self-comparison approach as he compared the student’s solution with his own solution. However, on in-depth reflection, it was noticed that this response should be a self-comparison approach not a mathematical approach because he based his explanation on his own working that he assumed to be correct. To overcome this limitation, I refined the definitions of all the approaches precisely before applying them to analyse the main study data.

In summary, the pilot study not only gave me the confidence to move forward with the main study but also increased the research quality. This is consistent with the claims made by past researchers that pilot testing enhances the reliability and validity in research (Gudmundsdottir & Brock-Utne, 2010; Malmqvist et al., 2019). By conducting the pilot study, I was better informed to undertake the main study through identification of weaknesses that may be addressed.

4.9 ETHICAL CONSIDERATIONS

The execution of this study followed the ethical guidelines of Queensland University of Technology (QUT). A Low Risk Ethical Clearance was approved by QUT for this study. The QUT ethics approval number for this study is 2021000063. The Malaysian Economic Planning Unit (EPU) and the head of Department of Mathematics of the targeted university also gave permission for the research to be conducted at that university. The approval letter from EPU is provided in Appendix D (p. 228).

In response to the COVID-19 pandemic, the data collection process followed tightly the advice provided by the Malaysian Ministry of Health, and thus conducted via Zoom. As discussed in Section 1.5 (p. 7), this study is completed in the midst of the pandemic. The methodology has undergone several iterations and corresponding refinements taking into consideration the advice from the faculty ethics advisor. An ethics variation was obtained to include an additional data collection tool to complement the change of method and to guarantee the contribution of the study (see Appendix E, p. 231).

As outlined in Section 4.4 (p. 60) about the sample of this study, all third year and final year students enrolled in the Bachelor of Education (Mathematics) (Hons) at the public university were invited to participate in the study ($n \approx 100$). On behalf of the researcher, the head of the Department of Mathematics of that university sent an invitation email to each of the targeted participants (Appendix F, p. 232). The approach email contained a participant information sheet (Appendix G, p. 233) and a consent form (Appendix H, p. 237). Students who were interested to participate in the study returned a completed consent form to the researcher via email. All the participating

PSTs gave consent for their written responses and audio-recordings of their voices to be used.

All the 11 PSTs gave consent to participate in this study. Since data were collected using Zoom, each research session was audio and video recorded as Zoom defaults to these combined modes. These recordings were stored securely in Cloudstor by default. Video recordings in Cloudstor were deleted immediately. Audio recordings were moved to the QUT Research Data Storage Service Folder then deleted from Cloudstor. As a backup, the data were also stored on the researcher's secure laptop which is password protected and the researcher's external hard drive which is also password protected. Viewing of all data is restricted to the researcher and the supervisors of this study.

In order to protect participants against "loss of dignity, self-esteem, privacy, or democratic freedoms" (Neuman, 2004, p. 47), the participation in this study is entirely voluntary. Concerning the fact of confidentiality protects participants' identity and reduce risks of sensitive issues being disclosed (Christians, 2005), pseudonyms are used when referring to consenting PSTs throughout the remainder of this study. The pseudonyms of the PSTs participants are outlined in **Table 4.1** (p. 61). In addition, a data management plan that safeguarded the collected data and the participants' identities was devised and implemented.

Chapter 5: Results

5.1 CHAPTER INTRODUCTION

The current study was designed to examine PSTs' mathematical pedagogical content knowledge of the order of operations. This chapter contains an analysis of the results from the questionnaire, task-based interviews, and lesson plans. Specifically, this chapter documents the results of PSTs' interpretation of mathematical expressions, their knowledge about mathematical connections between the order of operations and the properties of operations, their knowledge in determining the order of operations of contextualised problems, their interpretation of students' written work involving the order of operations, and their pedagogical knowledge of the order of operations. Note that some interview excerpts are included along with my analysis in the subsequent sections.

5.2 PRE-SERVICE TEACHERS' INTERPRETATION OF MATHEMATICAL EXPRESSIONS

This section reports the results of the first item on the questionnaire (Section 4.5.1, p. 63). This item consisted of eight mathematical expressions (Expressions 1a – 1h). It was designed to determine how PSTs use their knowledge of the order of operations to evaluate mathematical expressions. The PSTs' reasoning about the order they used in evaluating the expressions are also presented.

5.2.1 Expressions Involving Multiplication and Division

Expressions 1a and 1b were designed to analyse PSTs' knowledge about the left-to-right order for expressions involving multiplication and division. The expressions were posed as follows:

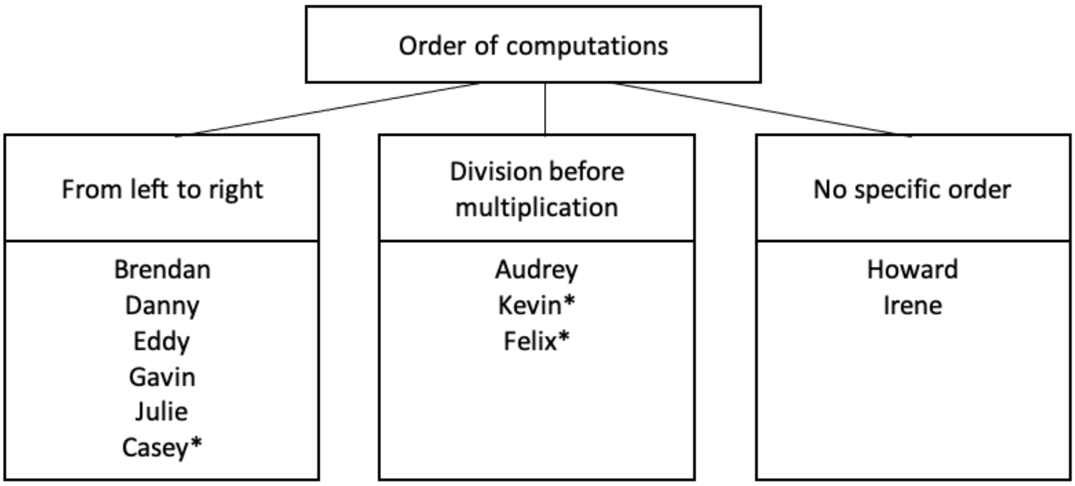
1a. $10 \div 5 \times 2$

1b. $4 \times 6 \div 3$

Figure *5.1* summarises the order the 11 PSTs used in evaluating the expressions.

Figure 5.1

PSTs' Order of Computations for Expressions 1a and 1b



Note. Participants who changed their responses during the interviews.

The correct order to evaluate both Expressions 1a and 1b is calculating from left to right. Of the 11 PSTs, six used the correct order, three used division before multiplication, and two executed the expressions without a specific order. Note that Casey, Kevin, and Felix changed their responses during the interviews and **Figure *5.1*** records their final decisions about the order used. The different orders of evaluation are discussed in turn.

5.2.1.1 From Left to Right

In evaluating the expressions with multiplication and division, six PSTs (Brendan, Danny, Eddy, Gavin, Julie, Casey) calculated from left to right. Brendan's response for Expression 1a is reproduced in **Figure *5.2*** and that for Expression 1b is reproduced in **Figure *5.3***. These responses illustrate the way that the six PSTs used the left-to-right order. The PSTs were required to give a reason for their ways of computation. Their reasons are discussed in turn.

Figure 5.2

Brendan's Response for Expression 1a

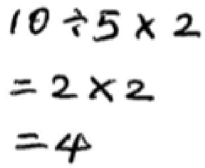
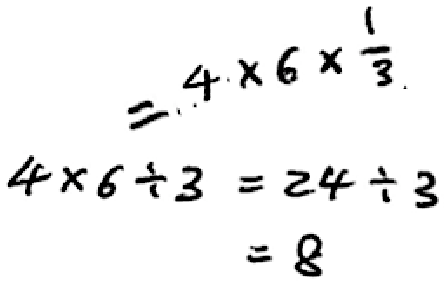


Figure 5.3

Brendan's Response for Expression 1b



The first reason that emerged from the data is related to inverse and associative properties. Brendan and Gavin explained why they used the left-to-right order based on the multiplicative inverse and associativity of multiplication. Brendan, for example, gave the explanation for Expression 1b as follow:

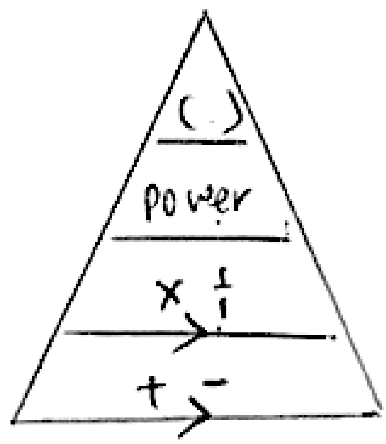
Brendan: Take Expression 1b for example, we can write division as multiplication or write multiplication as division. When they are all in multiplication, we can change their place, so any order doesn't matter. But when we maintain the question in multiplication and division, we have to follow the left-to-right order.

Writing $\div 3 = \times \frac{1}{3}$ implies that Brendan recognised division as the inverse of multiplication. He realised that the numbers in Expression 1b could be shifted if division was written as multiplicative inverse. Knowing that any order of operations does not affect the evaluation implies Brendan also based his explanations on the associativity of multiplication. He appeared to grasp the reason for why the left-to-right order worked based on the properties of multiplicative inverse and associativity of multiplication.

The second reason that emerged from the data is about another representation of the order of operations. Eddy explained his choice for evaluating the expressions from left to right by referring to the order of operations triangle. He drew the triangle to illustrate the hierarchy of operations (see **Figure *5.4***).

Figure 5.4

A Hierarchical Triangle That Shows the Order of Operations – Eddy's Response



Eddy's explanation was provided as follows:

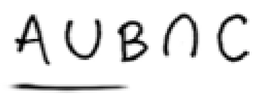
Eddy: We can solve questions using this triangle from top down. When we see brackets, we need to solve operations inside the brackets first. Next, we solve those with power. Then we solve multiplication and division and the lastly, we solve addition and subtraction. If the question contains only multiplication and division, we solve it from left to right. That's why I put an arrow at the line underneath multiplication and division, addition and subtraction.

Eddy's responses reveal that he recognised the order of operations based on a four-level hierarchical triangle (see **Figure *5.4***). He explained that the triangle has different levels of priority, with the highest priority at the top and the lowest priority at the bottom. Particularly, he used arrows to indicate the left-to-right order. This result suggests that Eddy interpreted the order of operations based on a hierarchical understanding of operations.

The third reason that emerged from the data is related to another mathematical concept. Julie explained why she used the left-to-right order based on set notation and set operations. She wrote $A \cup B \cap C$ as illustrated in **Figure *5.5***.

Figure 5.5

Use of Set Notation to Make Sense of the Order of Operations – Julie's Response



Julie's explanations were as follows:

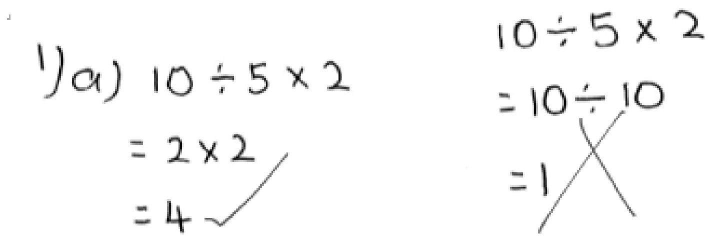
Julie: Just like the topic of set and subset. For mixed operations, we must do from left to right. For example, A union B and intersection C. We need to do it from the left first, which is the union first. I applied this concept to solve both Questions 1a and 1b.

The excerpt shows that the concept of set impacted Julie's interpretation of the left-to-right order. Based on her understanding about set notations and set operations, she performed the evaluation of expressions the same way as doing set operations. This result suggests that the learning of an idea at a more advanced stage may influence the way to explain an idea encountered earlier.

There was evidence that Casey and Danny gave no reason for their ways of computations. During the interview, it was observed that Casey changed her initial responses (responses provided in the questionnaire) from evaluating the expressions without a specific order to calculating from left to right. In this regard, the interviewer posed follow-up questions to gain a deeper understanding of Casey's responses and perspectives. It is important to note that the purpose of these follow-up questions was not to influence or change the student's minds, but rather to encourage Casey to provide more detailed and nuanced responses. By doing so, a comprehensive view of the student's thoughts and decision-making processes could be captured. Take Expression 1a for example. In the questionnaire, she provided two answers, that is, $10 \div 5 \times 2 = 2 \times 2 = 4$ and $10 \div 5 \times 2 = 10 \div 10 = 1$. In the interview, when she was required to explain her responses to this expression, she was uncertain about providing two different answers to an expression. After thinking through the possible orders of computations, she decided to evaluate Expressions 1a and 1b in order of appearance from left to right, and thus she changed her initial responses. As presented in **Figure *5.6***, she marked her first solution right and marked the second solution wrong.

Figure 5.6

Casey's Two Solutions for Expression 1a



Casey elaborated that, “It’s weird if I do from the back, and it’s also weird to have two answers. I think it should be from the front of the question.” Casey applied the correct order, but she could not provide a justification for why the order was correct. It is likely that Casey’s initial responses provided in the questionnaire were spontaneous and her responses given in the interview were reflective as she related to the impossibility of having two answers to one expression. It was her reflective thinking that made her used the correct order in evaluating the expressions.

5.2.1.2 Division before Multiplication

In simplifying the expressions involving multiplication and division, three PSTs (Audrey, Kevin, Felix) performed division before multiplication. For Expression 1b, these PSTs obtained the correct answer but used the wrong order. Audrey’s response, reproduced in **Figure *5.7***, illustrates the way that the three PSTs obtained the right answer for the wrong order.

Figure 5.7

Audrey’s Response for Expression 1b

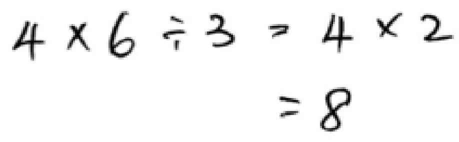


Figure *5.7* shows that Audrey first calculated 6 divided by 3 to get 2 then computed 4 times 2 to get 8. The answer 8 is correct but the order in evaluating the expression is incorrect.

All three PSTs explained their choice of using division before multiplication based on the acronym BODMAS. Audrey, for example, explained that:

Audrey: If I’m not wrong, BODMAS has been mentioned in a lecture in my first year but I’m not too sure. If I’m not wrong, following BODMAS, D division has to be calculated before M multiplication. DM means division then multiplication.

The excerpt reveals that Audrey misunderstood division precedes multiplication. This misunderstanding arose from the fact that she misinterpreted the acronym BODMAS based on the order in which the letters were presented. As the letter D comes before the letter M, Audrey perceived it as division takes precedence over multiplication. However, she referred to BODMAS with some degree of uncertainty as she expressed that “If I’m not wrong” and “I’m not too sure”.

Although Kevin and Felix also based their explanations on BODMAS, it is noted in the interviews that they changed their initial responses (responses provided in the questionnaire). Kevin changed from using the left-to-right order to prioritise division over multiplication. Felix changed from evaluating the expressions without a specific order to calculating division before multiplication. Similar to Audrey, both Kevin and Felix’s explanations were dependent on the acronym BODMAS. They had a literal understanding of the acronym that led them to an erroneous interpretation of the order of operations.

5.2.1.3 No Specific Order

In evaluating the expressions with multiplication and division, two PSTs (Howard, Irene) evaluated the expressions without a specific order. They provided two solutions to each expression. Howard’s response for Expression 1a is reproduced in **Figure *5.8*** and that for Expression 1b is reproduced in **Figure *5.9***. His responses illustrate the way that these two PSTs evaluated the expressions without giving priority to any operation.

Figure 5.8

Howard's Response for Expression 1a

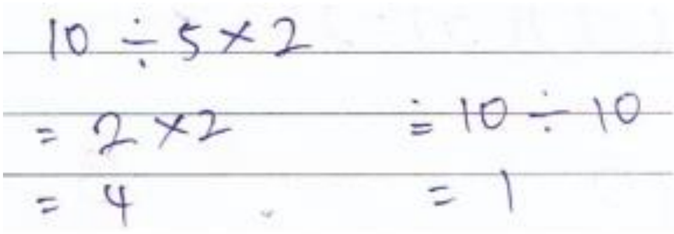
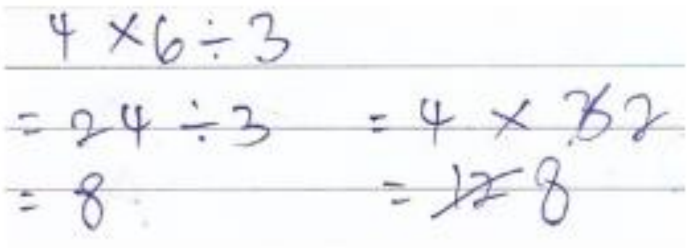


Figure 5.9

Howard's Response for Expression 1b

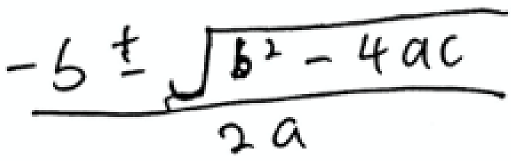


Both Howard and Irene referred to another concept learned in a higher level of mathematics to justify their ways of computation. Particularly, they referred to algebra in their responses. Howard, for example, stated that, "This kind of questions can have two answers. Just like an unknown, we can have two answers, for example x equals to 1 and -1, sometimes we can have 4 answers." Obtaining two answers from an unknown implies that Howard made sense of the order of operations based on his understanding that a quadratic equation can have two roots.

For Irene, she used the Quadratic Formula, as reproduced in **Figure *5.10***, to support her rationale for giving two answers to an expression.

Figure 5.10

Quadratic Formula Provided by Irene



Irene explained that, “When we use this formula, we normally get two answers. So, I think it is logic that the Question 1a has two answers.” Similar to Howard, Irene based her explanation on the quadratic formula that was used to find the roots of a quadratic equation. The responses of Howard and Irene indicate that they had an incorrect conception. They misinterpreted the order of operations based on their experience working with algebra, particular the quadratic equations. They did not recognise that a misapplication of the order can lead to different results.

To summarise this section, **Table 5.1** shows the explanations about the order of operations the PSTs used. Note that three PSTs offered no reason for the order of operations they used. Although six out of the 11 PSTs used the correct order to evaluate the expressions, only two of them could provide an accurate underlying reason for why the order is calculating from left to right. Knowing the reason is important for PSTs. When they know the reason, they can transfer their understandings and explain in ways that make sense to students. Moreover, knowing the reason allows them to be aware of the connections between mathematical ideas (Hatisaru, 2022).

Table 5.1

Explanations About the Order of Operations PSTs Used for Expressions With Multiplication and Division

Order of computations	Explanations of the order used
From left to right	Multiplicative inverse and associativity of multiplication Refer to other representations of the order of operations, that is, hierarchical triangle Refer to other mathematical concepts, that is, set
Division before multiplication	BODMAS
No specific order (provide two solutions and two different answers)	Refer to other mathematical concepts, that is, algebra

5.2.2 Expressions Involving Addition and Subtraction

Expressions 1c and 1d were used to determine PSTs' knowledge about the left-to-right order for expressions containing addition and subtraction. Both expressions were posed as follows:

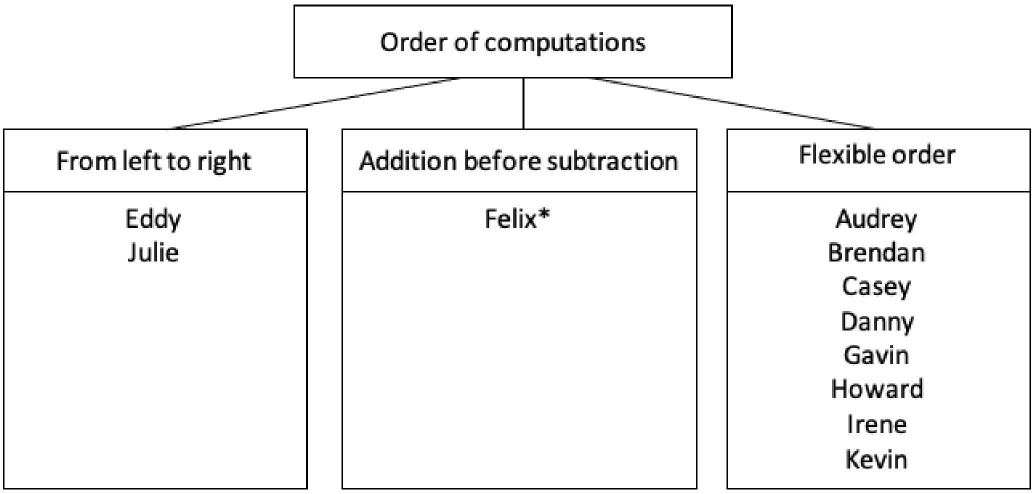
1c. $3 - 12 + 8$

1d. $4 + 17 - 7$

Figure *5.11* summarises the order the 11 PSTs used in simplifying the expressions.

Figure 5.11

PSTs' Order of Computations for Expressions 1c and 1d



Note. Participants who changed their responses during the interviews.

An analysis of the PSTs' responses reveals that all obtained the correct answer to each expression, but the PSTs differed in terms of their order of computations. Of the 11 PSTs, two computed the expressions from left to right, one proceeded with addition before subtraction, and eight executed the expressions in a flexible order. In this study, a flexible order refers to the PSTs' provision of two kinds of correct order of computation in getting the same final answer. Particularly, the order of evaluation does not matter for PSTs who used a flexible order because they started the evaluation from either addition or subtraction to arrive at the same final answer. A flexible order is different from no specific order, which was discussed in the Section 5.2.1.3 (p. 100). Using no specific order may result in two different answers but using a flexible order yields the same answer. Note that Felix changed his responses during the interview and **Figure *5.11*** records his final decisions about the way to evaluate the expressions.

5.2.2.1 From Left to Right

In evaluating the expressions with addition and subtraction, two PSTs (Eddy, Julie) calculated from left to right. Eddy's response for Expression 1c is reproduced in

Figure *5.12* and that for Expression 1d is reproduced in **Figure *5.13***. These responses illustrate the way that the two PSTs used the left-to-right order.

Figure 5.12

Eddy's Response for Expression 1c

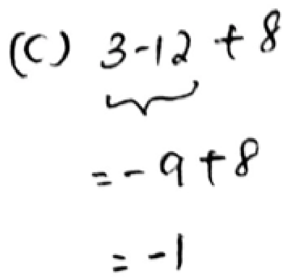
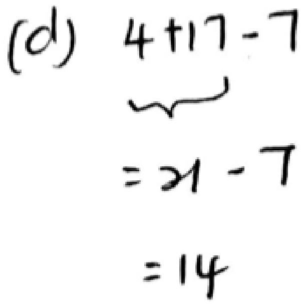


Figure 5.13

Eddy's Response for Expression 1d



When were required to give a reason for their ways of computation, Eddy again explained his choice of working from left to right by referring to another representation of the order of operations. He used a hierarchical triangle to explain his response (see **Figure *5.4***, p. 96). He placed an arrow at the lowest level of the hierarchical triangle to indicate that expressions involving addition and subtraction were to be evaluated from left to right. This implies that Eddy interpreted the order of operations based on a hierarchical understanding of operations. Julie, on the other hand, offered no reason for why she chose the left-to-right order.

5.2.2.2 Addition before Subtraction

In evaluating Expressions 1c and 1d, Felix used addition before subtraction. It is noted in the interview that Felix changed his initial responses (responses completed in the questionnaire). Initially in the questionnaire, he provided two solutions to each expression. However, during the interview, he chose to prioritise addition over subtraction. Felix's final decision for Expression 1c is reproduced in **Figure *5.14*** and that for Expression 1d is reproduced in **Figure *5.15***.

Figure 5.14

Felix's Response for Expression 1c

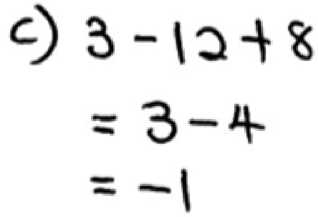
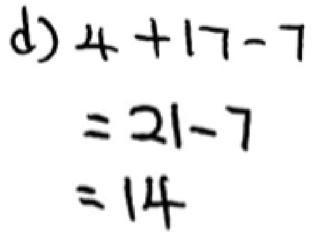


Figure 5.15

Felix's Response for Expression 1d



For Expression 1c, Felix first calculated $(-12) + 8$ to get -4. This implies that he presumed -12 given in the expression as $+(-12)$ based on his understanding that subtraction could be written as additive inverse. Using this interpretation, he could obtain the correct answer -1, but he based his explanation on a wrong order: he prioritised addition over subtraction. Although his written responses show that he was correct in evaluating both the expressions, his interview responses reveal that he had misunderstood the acronym BODMAS. He said that, “Because it has been stated in BODMAS that addition first subtraction later. If we do not follow the AS method, we might get a wrong answer.” This reasoning suggests that Felix used BODMAS incorrectly and believed the acronym required him to follow the sequence AS, that is, performing addition before subtraction. Despite having this misinterpretation about BODMAS, it is his interpretation of subtraction as the additive inverse that leads him to the correct answer for Expression 1c.

5.2.2.3 Flexible Order

In simplifying expressions with addition and subtraction, eight PSTs (Audrey, Brendan, Casey, Danny, Gavin, Howard, Irene, Kevin) evaluated the expressions in a flexible order. As mentioned earlier in this section, a flexible order refers to the PSTs’ provision of two kinds of correct order of computation in getting the same final answer. These PSTs realised that the order of computations does not matter because any order will eventually lead to the same answer. Casey’s response for Expression 1c is reproduced in **Figure *5.16*** and that for Expression 1d is reproduced in **Figure *5.17***. These responses illustrate the way that the eight PSTs used the flexible order.

Figure 5.16

Casey’s Response for Expression 1c

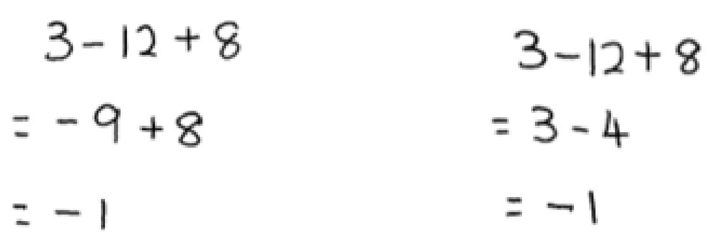
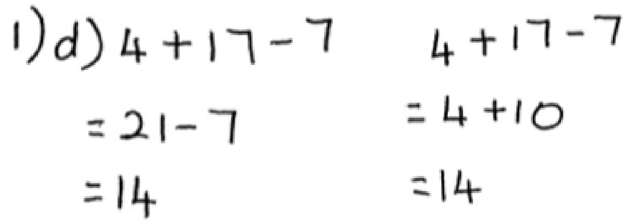


Figure 5.17

Casey's Response for Expression 1d

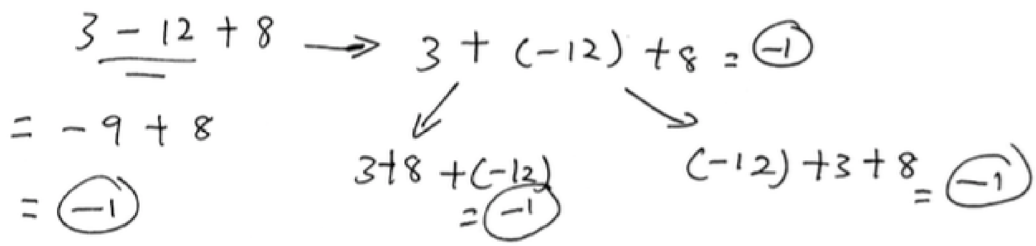


Casey explained that, "If I do subtraction first, I get -9 plus 8. If I do addition first, I get 3 minus 4. Both answers are the same so we can do either way." In her first solution to Expression 1c, Casey performed subtraction then addition. Contrary, in her second solution, she performed addition then subtraction by interpreting $3 - 12 + 8$ as $3 + (-12) + 8$. This shows that Casey had developed flexibility in simplifying this expression because she recognised that the final answer would remain the same regardless of which operation had a priority.

When the PSTs were required to give a reason for using a flexible order, their explanations were based on the associativity of addition, and they made sense of subtraction as additive inverse. Gavin, for example, explained his response through manipulating the expression $3 - 12 + 8$ (see **Figure *5.18***).

Figure 5.18

Explanation for Why the Left-to-right Order Works – Gavin's Response



As in **Figure *5.18***, Gavin first interpreted subtraction as additive inverse so he rewrote the expression $3 - 12 + 8$ into $3 + (-12) + 8$. From there, he drew on his recognition of associativity of addition to rearrange the numbers in different combinations. He explained:

Gavin: I can play around the numbers either become 3 plus 8 plus -12 or -12 plus 3 plus 8. I will still get the answer -1. But if we look back to the question, if I didn't change to -12, 12 plus 8 is 20. 3 minus 20 is -17 which is not the same as -1. So, it means that we need to do from left to right only then we can get the correct answer -1.

According to the excerpt, Gavin associated the three numbers in the expression in any way he desired. He appeared to have the knowledge of associativity and managed to apply this knowledge when making sense of the order of operations. He also highlighted the need to interpret subtraction as additive inverse by explaining that $3 - 12 + 8$ might be perceived wrongly as $3 - (12 + 8)$ that eventually generated an erroneous answer -17. He thus affirmed that operations have to be computed from left to right. Gavin recognised not only a flexible order but also appeared to grasp why the left-to-right order worked for expressions with addition and subtraction. This result shows that it is his interpretation of subtraction as the additive inverse and his understanding about associativity that assisted him in making sense of the left-to-right order.

Kevin also made sense of subtraction as additive inverse, although this was not explicit. His responses, however, demonstrated a misunderstanding about the acronym BODMAS. He stated that, "I am confused now. BODMAS is AS, addition then subtraction, but I don't think this is correct. I used to do questions like this from left to right." As he evaluated expressions involving addition and subtraction commonly from left to right, he viewed BODMAS as contradictory to his usual method. However, while he was trying to make sense of the contradiction, he suddenly realised that BODMAS was still valid. His further explained:

Kevin: Ahh, I know! I know why BODMAS is written as AS. We get the same answer no matter we do addition first or subtraction first. If you don't believe, you try 1c and 1d. The answer will be the same. If I do 1c as 3 minus 4, it is -1, if I do it as -9 plus 8, it is also -1. BODMAS said addition first then subtraction, there is nothing wrong with BODMAS because if I do addition first the answer is still correct. If someone do subtraction first, he or she also not wrong because the answer will also be the same.

For Kevin, the letters AS in BODMAS was justifiable as he drew on the fact that either way of evaluating the expression yielded the same final answer. His claim, indeed, was made on the basis of his understanding about subtraction as the additive inverse although this was not explicitly explained. However, the excerpt demonstrates a misunderstanding Kevin possessed that arose from the fact that he misinterpreted BODMAS based on the order in which the letters were presented.

To summarise this section, **Table 5.2** shows the explanations about the order of operations the PSTs used. Note that only Julie offered no reason for the order of operations she used. All the 11 PSTs obtained the correct answer when evaluating Expressions 1c and 1d. However, Felix obtained the correct answer for the wrong interpretation. He gave priority to addition over subtraction based on a literal understanding about BODMAS. This revealed the danger of using acronyms in the order of operations.

Table 5.2

Explanations About the Order of Operations PSTs Used for Expressions With Addition and Subtraction

Order of computations	Explanations of the order used
From left to right	Refer to other representations of the order of operations, that is, hierarchical triangle
Flexible order (provide two solutions with the same answer)	Additive inverse and associativity of addition
Addition before subtraction	BODMAS

On the other hand, building on their understanding about subtraction was equivalent to the additive inverse and addition was associative, most of the PSTs had developed the flexibility in evaluating expressions containing addition and subtraction. This means they could perform calculations involving addition and subtraction using any orders without affecting the final answer. The analysis differentiates how the PSTs made sense of the left-to-right order for addition and subtraction as well as for division and multiplication. With the knowledge about associativity of addition and subtraction as additive inverse, the PSTs could interpret the left-to-right order for addition and subtraction and perform calculations correctly. Having this flexibility is essential for PSTs in helping students to make sense of the rule.

5.2.3 Expressions Involving Indices

Expressions 1e and 1f were formulated to analyse the PSTs' knowledge about operating on indices when executing the order of operations. The expressions were posed as follows:

1e. $10 - 3^2$

1f. 2^{3^2}

The results to each expression are discussed in turn.

5.2.3.1 An Expression with Exponentiation and Subtraction

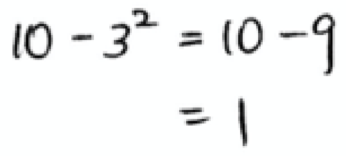
The use of subtraction in the Expression 1e $(10 - 3^2)$ was designed to determine if the PSTs knew that the index 2 had the base of 3, but not -3. Their responses were also analysed in terms of whether they used subtraction first or obtained the value of the index first. The correct order to evaluate this expression is operating on the index prior to subtraction. All PSTs used the correct order of operations to evaluate this expression.

Exponentiation Before Subtraction

In evaluating the Expression 1e, all the PSTs $(n = 11)$ operated on the index before doing subtraction. They squared 3 before subtracting it to get the answer 1. Audrey's written response to this expression, reproduced in **Figure *5.19***, illustrates the way that all the PSTs used the correct order to evaluate the expression.

Figure 5.19

Audrey's Response for Expression 1e

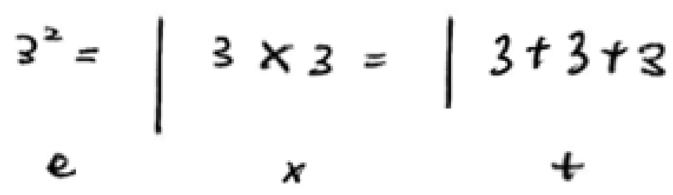


All the PSTs responded in the interviews that simplifying indices before doing any of the four basic operations was the correct sequence for evaluating mathematical expressions. They were required to give a reason for simplifying indices before subtraction. These reasons are discussed in turn.

The first reason that emerged from the data is that indices mean repeated multiplication. Brendan explained that priority was given to indices because indices could mean repeated multiplication. He provided numerical examples to support his explanation (see **Figure *5.20***).

Figure 5.20

Examples to Illustrate the Order of Operations – Brendan's Response



Brendan's explanations were as follows:

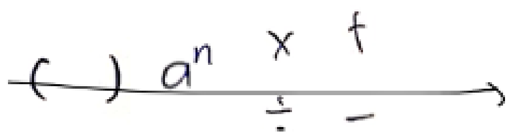
Brendan:	Starting with addition, which is at the first level. Take for example, 3 plus 3 plus 3. We can write 3 plus 3 plus 3 equals to 3 times 3 because they are equivalent. 3 times 3 is at the second level and it involves multiplication. Then, we can write 3 times 3 equals to 3 squared because they are also equivalent. This square is an exponent and is one level higher than multiplication.
Researcher:	Why do you prioritise the exponent before doing subtraction?
Brendan:	It's because subtraction is at the same level with addition.

Brendan's responses indicate that he viewed addition, multiplication, and indices as three interrelated concepts. He explained the connectedness of the operations based on a hierarchical understanding, ranging from addition to multiplication to exponents. For him, multiplication takes precedence over addition because multiplication is adding equal groups together, whereas exponentiation takes precedence over multiplication because exponentiation is multiplying equal groups together. Since he recognised that subtraction has the same priority as addition, he presumed exponentiation takes precedence over subtraction.

The second reason that emerged from the data is about another representation of the order of operations. Danny explained his choice of prioritising indices by using symbols to illustrate the order of operations (see **Figure *5.21***).

Figure 5.21

Use of Symbols to Illustrate the Order of Operations – Danny's Response



Danny explained that, "We need to simplify exponents first because from brackets then exponents, multiplication and division, and lastly addition and subtraction." He used an arrow to illustrate the operation sequence ranging from parentheses to addition and subtraction. He explained that any index needs to be evaluated before the four basic operations. This result shows that Danny recognised the proper hierarchy of the order of operations and applied the order correctly when he was presented with the expression that contains exponentiation and subtraction.

The third reason that emerged from the data is related to the acronym BODMAS. Felix, Gavin, and Kevin mentioned the acronym with statements like "By referring to BODMAS, O is power that comes before division, then MA and last one S. S is subtraction which is the last operation to calculate." For these PSTs, O is Order of Powers or Roots. They interpreted BODMAS based on the ordered letters of the acronym. As the letter O comes before the letter S, they perceived it as exponents takes precedence over subtraction.

The fourth reason that emerged from the data is related to a strategy used in simplifying expressions. Casey, Eddy, and Irene operated on indices first because of the need to obtain a simpler form of an expression. Casey, for example, stated that, "Because doing power first will result in a simpler form that contains small numbers," and Eddy explained that, "An exponent means the number of times a number is multiplied by the number itself, which is a bit complicated, so we can reduce this complex idea to a simple idea." These excerpts show that the PSTs presumed indices as a complex concept that needed to be simplified prior to other operations, that is, subtraction.

The last reason that emerged from the data is related to the use of examples. Julie explained why she used exponentiation before subtraction by viewing indices as area. Her explanation was as follows:

Julie: We used to calculate area using exponents. We calculate the area of a living room before we can minus that area from the area of a whole house. Considering the 3 squared is an area, we need to find 3 times 3 first before minus the answer by 10.

The except implies that Julie used a common real-world application of indices, which is the area concept, to interpret the order of operations for Expression 1e. Her interpretation about index as area is acceptable only in this particular square example, but it cannot be generalised beyond the example.

To summarise this section, **Table 5.3** shows the explanations about the order of operations the PSTs used in evaluating the expression with exponentiation and subtraction. Note that Audrey, Danny, and Howard offered no reason for the order of operations they used. Although the results show that all the PSTs could get the correct answer for Expression 1e, most of them were based on the wrong reasons. Only Brendan recognised the underlying reasoning that exponentiation takes precedence based on the idea of conceiving indices as repeated multiplication. Getting a correct answer is insufficient because seeing the underlying reasons may help in organising the mathematical knowledge into a coherent whole through making relevant connections between different concepts (Toh & Choy, 2021).

Table 5.3

Explanations About the Order of Operations PSTs Used for Expressions With Exponentiation and Subtraction

Order of computations	Explanations of the order used
Exponentiation before subtraction	Repeated multiplication Refer to other representations of the order of operations, that is, symbols in hierarchical form BODMAS A strategy in getting a simpler form of an expression Use of examples

5.2.3.2 An Expression with Stacked Exponents

The Expression 1f $\left(2^{3^2}\right)$ involves stacked exponents. **Figure *5.22*** summarises the order the PSTs used in evaluating this expression. To evaluate Expression 1f, the correct order is to work from right to left. Of the 11 PSTs, six used the correct order, three used the left-to-right order, and two computed multiplication before exponentiation. The different orders of evaluation are discussed in turn.

From Right to Left

In evaluating Expression 1f, six PSTs (Brendan, Eddy, Felix, Gavin, Howard, Kevin) calculated from right to left. This order may be known as the top-down order. Eddy’s written response to this expression, reproduced in **Figure *5.23***, illustrates the way that the six PSTs worked from the right to the left when evaluating the expression. The PSTs were required to give a reason for their ways of computation.

Figure 5.22

PSTs' Order of Computation for Expression 1f

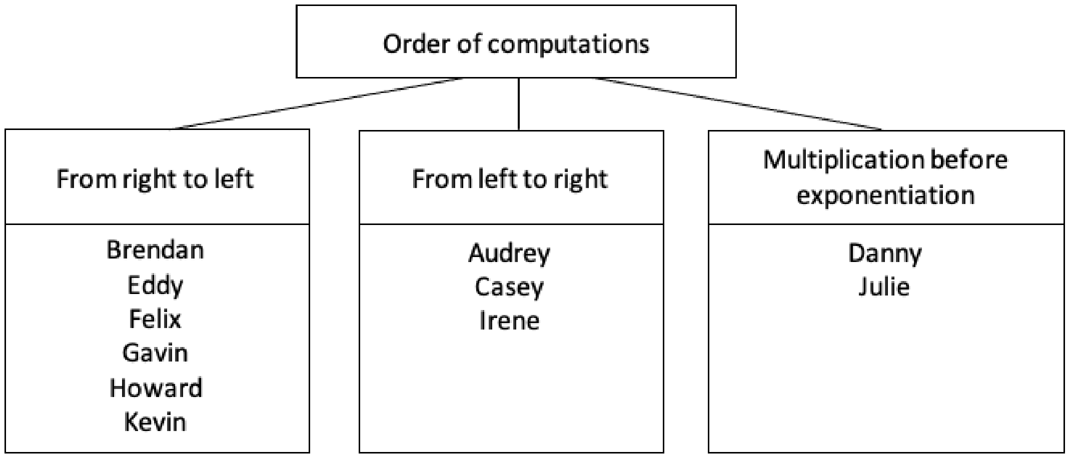
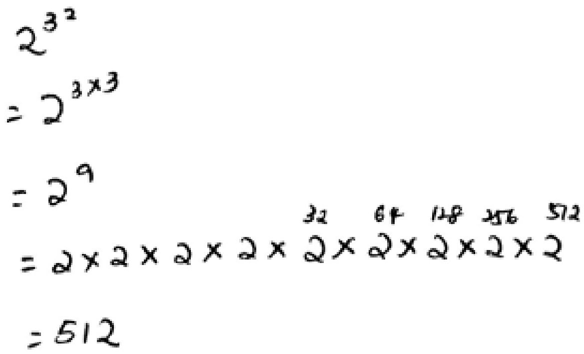


Figure 5.23

Eddy's Response for Expression 1f



Eddy first evaluated 3^2 to get 9 then simplified 2^9 to obtain the answer 512. He applied the right-to-left order based on the definition of indices. His explanations were as follows:

Eddy: As I see here, 2 is the base and 3 squared is the power. We cannot find 2 cubed first because the 3 is not a complete number, the 3 has a square as its power, so we need to find what is 3 squared first.

Researcher: What makes you think that 2 is the base but not 2 cubed?

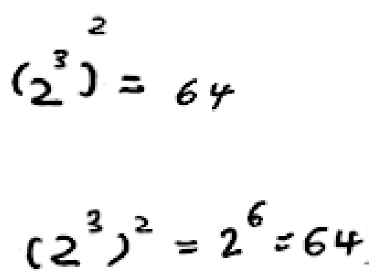
Eddy: We know that, for example, 4 to the power of 3 means 4 times 4 for 3 times. In this question, 2 to the power of 3 to the power of 2 means 2 is multiplied by 2 itself for 3 squared times. We don't know what is 3 squared times unless we find out the answer for 3 squared first. So, 3 squared is 9. Now we know 2 is multiplied by 2 for 9 times, which is 512.

The excerpt indicates that Eddy read 2^{3^2} as $2^{(3^2)}$ and viewed its exponents (3^2) as an expression that needed to be simplified first. The result shows that Eddy made sense of 2^{3^2} by relating the stacked exponents to the number of times the base needed to be multiplied. His interpretation leads him to arrive at the correct answer 512.

On the other hand, Brendan and Gavin used a correct order but based on an incorrect reason when evaluating Expression 1f. They compared the stack exponents with the Power Rule of exponents. Brendan, for example, stated that, "If we do 2 to the power of 3 first, it is actually the same with 2 to the power of 3 times 2. But this question is different from 2 to the power of 3 times 2." He used $(2^3)^2$ to further support his reasoning (see **Figure *5.24***).

Figure 5.24

Comparing Stacked Exponents With the Power Rule – Brendan's Response



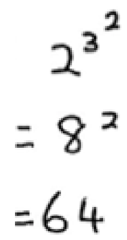
Brendan's responses suggest that he interpreted $(2^3)^{\,2} = (2^3)^2$ because both yield the same answer 64. His verbal response also indicates that he realised 2^{3^2} could not mean $(2^3)^2$. In other words, he presumed 2^{3^2} to mean $2^{(3^2)}$. Although the excerpts show that Brendan understood stacked exponents and the Power Rule, he did not explain why he evaluated the index first.

From Left to Right

To evaluate the Expression 1f, three PSTs (Audrey, Casey, Irene) worked from left to right, which is an incorrect order. They first calculated 2^3 to get 8 then simplified 8^2 to yield the erroneous answer 64. Casey's written response to this expression, reproduced in **Figure *5.25***, illustrates the way that the three PSTs worked from left to right.

Figure 5.25

Casey's Response for Expression 1f



Casey referred to the natural way of reading from left to right to justify her order of computation. Her explanation was as follows:

Casey: Let say we have two additions in an expression, 1 plus 2 plus 3. To do this question, we follow the way we read the question. We do 1 plus 2 first, then 3 plus 3, the answer is 6. Same as 5 minus 3 minus 1 or 2 times 3 times 4. So, in this question we have two powers. I think it is correct to start doing the one we read first. Since we read 2 to the power of 3 first, we find the answer for 2 to the power of 3 first. After that, we find 8 to the power of 2 so that the answer is 64.

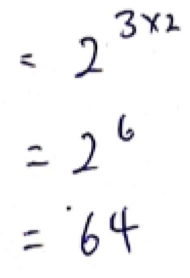
This excerpt implies that Casey made sense of the order of precedence for stacked exponents based on the natural way of reading from left to right. She used a few examples to support her explanation and interpreted them correctly. For her, an expression that contains a repeating operation must be evaluated from left to right. This incorrect perception was not applicable to expressions with stacked exponents and eventually led Casey to deviate from the correct order.

Multiplication Before Exponentiation

In simplifying the expression with stacked exponents, Danny and Julie performed multiplication before exponentiation. They first multiplied the stacked exponents, 3 and 2, to get 6 then calculated 2^6 to obtain the erroneous answer 64. Danny's written response to this expression, reproduced in **Figure *5.26***, illustrates the way that the two PSTs calculated multiplication prior to exponentiation.

Figure 5.26

Danny's Response for Expression 1f



The Power Rule of exponents surfaced in Danny and Julie's explanation for performing multiplication before exponentiation. Both of them mentioned the rule explicitly, with Danny stating that, "I use the power law of indices to multiply 3 and 2," and Julie explaining that, "I'm following the law of index when we have power of power, we need to multiply the powers first." This result suggests that Danny and Julie presumed the expression 2^{3^2} to mean raising a number with an exponent to a power. They viewed 2^{3^2} as $(2^3)^{^2}$. From there, they considered $(2^3)^{^2} = (2^3)^2$ and applied the Power Rule of exponents inaccurately.

To summarise this section, **Table 5.4** shows the explanations about the order of operations the PSTs used in evaluating the expression with stacked exponents. Six PSTs used the correct order (right to left) to evaluate Expression 1f but five employed incorrect orders. Note that Audrey, Felix, Howard, Irene, and Kevin offered no reason for the order of operations they used. The PSTs had used the incorrect order or were unable to explain the order they used, which is mostly likely because expressions with stacked exponents were not common in school mathematics, so such expressions were unfamiliar to them.

Table 5.4

Explanations About the Order of Operations PSTs Used for Expressions With Stacked Exponents

Order of computations	Explanations of the order used
From right to left	Definition of indices Power Rule of exponents
From left to right	Refer to natural way of reading
Multiplication before exponentiation	Power Rule of exponents

5.2.4 Expressions Involving Parentheses

Expressions 1g and 1h were designed to examine the PSTs' knowledge about the use of parentheses when executing the order of operations. The expressions were posed as follows:

1g. $8 - ((-2) + 4)$

1h. $(9 - 2)(7 - 4)$

The correct order to evaluate these expressions is executing the operations inside the parentheses before performing other operations.

5.2.4.1 Parentheses First

All the 11 PSTs simplified the Expressions 1g and 1h using the correct order of operations, that is, parentheses first. For Expression 1g, they subtracted the sum of -2 and 4 from 8 to obtain the answer 6. For Expression 1h, they first calculated $9 - 2$ and $7 - 4$ then multiplied the answers to get 21. Kevin's response for Expression 1g is reproduced in **Figure *5.27*** and that for Expression 1h is reproduced in **Figure *5.28***. These responses illustrate the way that the PSTs used the correct order to evaluate the expressions. The PSTs were required to give a reason for their ways of computation. These reasons are discussed in turn.

Figure 5.27

Kevin's Response for Expression 1g

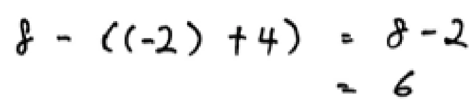
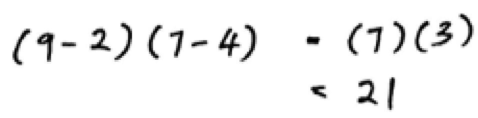


Figure 5.28

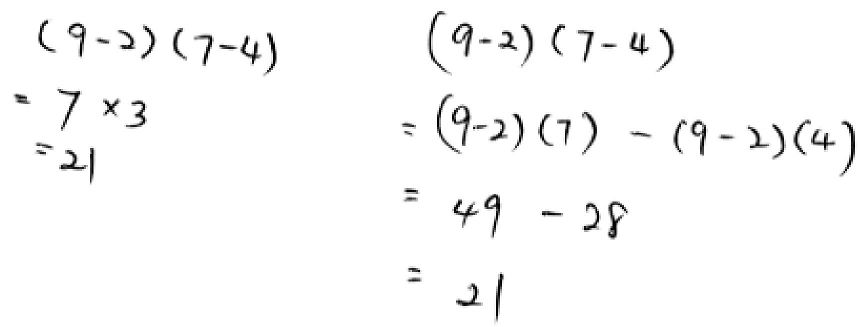
Kevin's Response for Expression 1h



The first reason that emerged from the data is related to the distributive property. Audrey compared the computations by using order of operations and the distributive property (see **Figure *5.29***).

Figure 5.29

Comparing the Order of Operations and the Distributive Property for Expression 1h – Audrey's Response



Audrey's explanation was as follow:

Audrey: We can write the question in another way. For example, $a \times (b + c)$. In Question 1h, we can write it to become $(9 - 2)(7) - (9 - 2)(4)$. The answer is also 21. This shows that 21 is the correct answer and doing bracket first is the correct way.

Rewriting $(9 - 2)(7 - 4)$ into $(9 - 2)(7) - (9 - 2)(4)$ implies that Audrey based her evaluation of the expression on the distributive property. Since the same answer 21 was obtained, she explained that parentheses must be given priority. This result shows that Audrey made sense of the order of operations by making connections to the distributive property. Making connections between the order of operations and the distributive property eventually led Audrey to give priority to parentheses.

The second reason that emerged from the data is based on another representation of the order of operations. Eddy again explained his choice of prioritising parentheses based on the hierarchical triangle of the order of operations (see **Figure *5.4***, p. 96). He explained that, "When we see brackets, we must solve what is inside the brackets first because it is at the top of the pyramid." For Eddy, parentheses that is at the top of the triangle indicates the top priority must be given to parentheses. This finding reveals that Eddy interpreted the order of operations hierarchically using a graphical display, that is the order of operations triangle.

The third reason that emerged from the data is related to the acronym BODMAS. Felix, Gavin, and Kevin referred to BODMAS and mentioned explicitly that the letter B indicates the top priority for order of operations. Felix, for example, stated that, "B for brackets is the first letter in BODMAS that's why whatever operation is inside a bracket, it is solved first." Gavin also mentioned BODMAS repeatedly to emphasise the need of prioritising any operation insides the parentheses. He explained that, "It is again BODMAS. Every question is related to BODMAS and involves is brackets, so B the first letter of BODMAS tells us that brackets are the first thing to do." The findings suggest that the PSTs interpreted the order of operations literally based on the acronym.

The last reason that emerged from the data is related to the use of examples. Julie conceived of two different examples to make sense of how to handle parentheses. Julie's explanation was provided as follows:

Julie: The first situation will be like 8 times bracket 3 plus 2. This means adding 3 and 2 then multiply the result by 8. Bracket is a must in this example to tell addition must be started first. For example, one box has 3 red marbles and 2 yellow marbles, there are 8 similar boxes. The second situation can be like 8 times 3 plus 2 without any bracket. This means multiplying 8 by 3 then add the product with 2. Bracket is not needed here. For example, one box has 3 marbles, there are 8 similar boxes, then there are 2 additional marbles which are not in those boxes.

In the first instance $8 \times (3 + 2)$ given by Julie, there was a clear indication that she perceived the use of parentheses as a way to specify an operation that requires primary attention. In the second instance $8 \times 3 + 2$, her response suggests that multiplication preceded addition. Julie reasoned that parentheses have the highest priority through using examples.

Note that there was evidence that the PSTs used emphasising brackets as a strategy to help them use the order of operations. To show the operation they first executed, Danny, Eddy, Gavin, and Julie added parentheses in their solutions for Expressions 1a and 1b whereas Eddy and Julie added parentheses for Expression 1c and 1d. Danny's response for Expression 1a is reproduced in **Figure *5.30*** and that for Expression 1b is reproduced in **Figure *5.31***. These responses illustrate the way that the PSTs used emphasising brackets when evaluating the expressions.

Figure 5.30

Danny's Response for Expression 1a

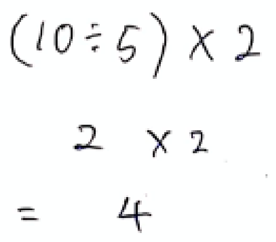
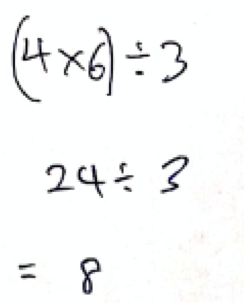


Figure 5.31

Danny's Response for Expression 1b



Instead of using parentheses, Eddy used underbraces ($\underbrace{\quad}$) in his solutions to mean the first operation he computed. His response for Expression 1a is reproduced in **Figure *5.32*** and that for Expression 1b is reproduced in **Figure *5.33***. It is seen that he also changed the expressions into fraction form to assist his computations.

Figure 5.32

Eddy's Response for Expression 1a

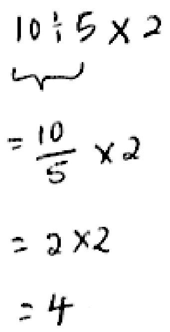
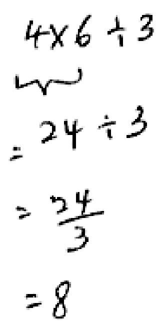


Figure 5.33

Eddy's Response for Expression 1b



The reason the PSTs provided for adding parentheses was that the emphasising brackets facilitated their computations. Gavin, for example, elaborated that, "I just want to make it step by step by showing the first part of my solution." Eddy also explained that, "I put the symbol for the first solution that I need to do, they help me to see easily what I need to do." This finding suggests that emphasising brackets was a visual representation of the PSTs' knowledge of the order of operations. They relied on the emphasising brackets as a strategy that helped them use the order of operations.

To summarise this section, **Table 5.5** shows the explanations about the order of operations the PSTs used in evaluating the expression involving parentheses. All the PSTs used a correct order of operations to evaluate the expressions. A potential explanation for this might be that the PSTs were accustomed to do any operations inside of parentheses as the first step. Moreover, the analysis also revealed that the PSTs added emphasising brackets to some expressions to facilitate their way of evaluation, particularly in visualising the first step to be carried out. However, six of the PSTs offered no reason for why they gave priority to parentheses. It may be that the use of parentheses in relation to the order of operations is an arbitrary convention and the PSTs had not ever considered the reason for this convention.

Table 5.5

Explanations About the Order of Operations PSTs Used for Expressions With Parentheses

Order of computations	Explanations of the order used
Parentheses first	Distributive property
	Refer to other representations of the order of operations, that is, hierarchical triangle
	BODMAS
	Use of examples

5.3 PRE-SERVICE TEACHERS' KNOWLEDGE ABOUT THE CONNECTIONS BETWEEN THE ORDER OF OPERATIONS AND THE PROPERTIES OF OPERATIONS

Item 3 of the questionnaire (Section 4.5.1, p. 63) was designed to examine PSTs' knowledge about the connections between the order of operations and the properties of operations. This item consisted of two hypothetical situations (Items 3a and 3b). The results of each of the hypothetical situations are discussed in turn.

The goal of the first hypothetical situation was to explore how PSTs relate the left-to-right order to the associative property of addition and the additive inverse. The first hypothetical situation is shown in **Figure *5.34***.

Figure 5.34

Item 3a of the Questionnaire

3) a) Lisa and Richard evaluate the expression $4 + 3 - 2$ differently. They ask you whose solution is correct.

Lisa's response	Richard's response
$4 + 3 - 2 = 7 - 2$ $= 5$	$4 + 3 - 2 = 4 + 1$ $= 5$

(i) Assess the students' responses.

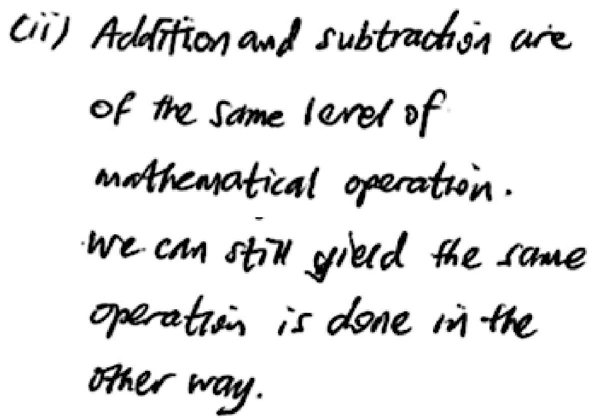
(ii) Explain why Lisa and Richard can obtain the same answer even though they evaluate the expression in different ways.

This item requires the knowledge of the order of operations, the associative property of addition, and the additive inverse. Of the 11 PSTs, six (Audrey, Brendan, Casey, Danny, Gavin, Irene) responded that both Lisa and Richard were correct whereas five (Eddy, Felix, Howard, Julie, Kevin) regarded Lisa as correct.

Responses to Item 3a(ii) were analysed to determine how PSTs recognised the connections between the left-to-right order and the properties of operations. Three explanations emerged from this analysis. In the first explanation, Audrey, Brendan, and Gavin used the additive inverse and the associative property of addition. Brendan's response is indicative of this type of explanation and his written response is reproduced in **Figure *5.35***.

Figure 5.35

Brendan's Response for Item 3a(ii)



Brendan explained in the interview that:

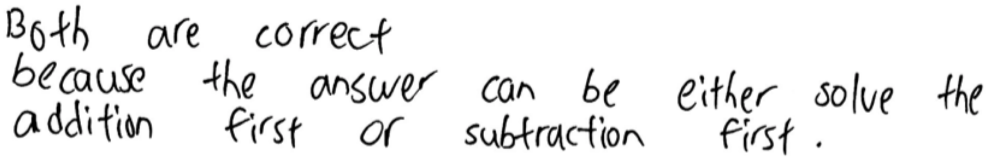
Brendan:	I would say this question can be done in any way you desired because adding up three numbers can be done in either way, the sum will not be affected. For example, $a + b + c$ or $b + c + a$ or $c + b + a$ and others, we still can get the same final sum.
Researcher:	But the question here involves minus 2, not all about addition.
Brendan:	This is because we can write the minus to become plus, to become $4 + 3 + (-2)$. Writing in this way means all addition so doing whichever will not create a problem.

Brendan's response shows that he could generalise the associativity. He noticed that calculations could be performed in different orders for $4 + 3 + (-2)$ and the final sum remained the same. He responded that $3 - 2$ could be written as $3 + (-2)$ because he interpreted subtraction as the opposite of addition. The finding shows that Brendan interpreted the left-to-right order by attending to additive inverse and the associative property of addition. He could apply his knowledge flexibly in order to make sense of the order of operations.

In the second explanation, Casey, Danny, and Irene asserted that any order was acceptable as long as the answer was the same. Irene's response is indicative of this type of explanation and her written response is reproduced in **Figure *5.36***.

Figure 5.36

Irene's Response for Item 3a

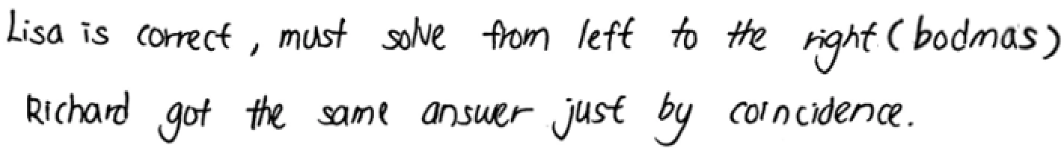

Both are correct
because the answer can be either solve the
addition first or subtraction first.

Irene stated that, "If we do the correct steps and get exactly the same answer, either way is okay." This response implies that Irene focused on the result of the computations. A potential explanation for this procedural application might be that Irene had no idea about the reason for why the order does not matter. She made connections only based on the similarities of the two orders.

In the third explanation, Felix, Julie, and Kevin referenced to BODMAS. Kevin's response is indicative of this type of explanation and his written response is reproduced in **Figure *5.37***.

Figure 5.37

Kevin's Response for Item 3a(ii)


Lisa is correct, must solve from left to the right (bodmas)
Richard got the same answer just by coincidence.

Kevin explained that, "BODMAS tells us to do in the order the question is presented and Richard got the answer just by coincidence." He linked the order to the mnemonic used to memorise the procedures. Although he was able to connect the order conventions to another representation, he appeared to be lacking the knowledge to relate the order of operations to the properties of operations.

The second hypothetical situation was designed to examine how PSTs relate the order of operations to the distributive property. The second hypothetical situation is shown in **Figure *5.38***.

Figure 5.38

Item 3b of the Questionnaire

3 b) Olivia and Desmond evaluate the expression $4(2+3)$ differently. They ask you whose solution is correct.

Olivia's response	Desmond's response
$4(2+3) = 4(5)$ $= 20$	$4(2+3) = 4(2) + 4(3)$ $= 8 + 12$ $= 20$

(i) Assess the students' responses.

(ii) Explain why Olivia and Desmond can obtain the same answer even though they evaluate the expression in different ways.

This item requires the knowledge of the order of operations and distributive property. An analysis of the PSTs' responses to Item 3b revealed that nine regarded both Olivia and Desmond as correct whereas two (Audrey, Felix) responded that Olivia was correct.

Responses to Item 3b(ii) was analysed to determine how PSTs recognised the connections between the order of operations and the distributive property. Five PSTs (Audrey, Brendan, Gavin, Irene, Kevin) explained that both the students' responses followed the order of operations, but Desmond distributed the number 4 before following the order. Gavin's response is indicative of this type of explanation.

Gavin: Desmond used distributive property to distribute the term outside the bracket with the terms inside the bracket, which is usually in the form of $a(b + c)$ to be distributed to $ab + ac$. In this stage, order does not matter as we only rewrite or distribute the terms. After distributing, now the order does matter where the BODMAS order must take place.

Gavin's response reveals that he made connections between the order conventions and the distributive property by referring to their properties. He regarded the distributive property as a way to rewrite the expression before using the order of operations to determine the sequence of computations. Furthermore, he could generalise the distributive property.

Audrey also explained the situation based on the distributive property, but she had limited understanding about the distributive property. She gave an algebraic expression to explain her view as follows:

Audrey: Desmond is correct if he puts an unknown inside, so in this question, Desmond is wrong, Olivia is correct. We used to do like Desmond if there is an unknown like x or y inside the bracket. For example, maybe $4(x + 3)$. So, Desmond is wrong because his method is for questions with unknown. Olivia is correct because she follows the order to solve what is in the bracket first.

Audrey considered Desmond's method of distributing 4 to each term in the parentheses as incorrect because she believed that this method was applicable only for expressions with variables. She perceived the order of operations as the correct method to evaluate the expression. In fact, Audrey's perception is inaccurate because the distributive property holds true for multiplying a given number by the sum of two numbers, not restricted to variables. The finding suggests that Audrey understood how to use both the order conventions and the distributive property but the connection she made contained an inaccurate perception.

To summarise this section, the PSTs' explanations centred mainly around subtraction is additive inverse and addition is associative when interpreting the left-to-right order. There were also PSTs who made sense of the order based on the mnemonic BODMAS and the same final answer. When linking to the distributive property, the PSTs viewed the distributive property as a way to rewrite the expression that subsequently allowed them to use the order of operations to simplify the expression.

5.4 PRE-SERVICE TEACHERS' KNOWLEDGE IN DETERMINING THE ORDER OF OPERATIONS OF CONTEXTUALISED PROBLEMS

This section reports the results of Item 2 of the questionnaire (Section 4.5.1, p. 63) concerning how the PSTs determined the order of operations when mathematising contextualised problems. This item comprised four contextualised problems (Problems 2a – 2d). The PSTs were required to write a single mathematical expression to best represent each of the four contextualised problems. The PSTs were then interviewed to explore the reasons that underpinned their given expressions. The results of each of the contextualised problems are discussed consecutively.

The first contextualised problem (Problem 2a) involved multiplication and division. This contextualised problem was posed as follows:

> *A tyre factory produces 6351 tyres every 3 days. How many tyres will the factory produce in 14 days?*

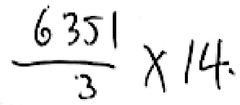
Problem 2a can be represented using the expression $6351 \div 3 \times 14$ or $\frac{6351}{3} \times 14$, in which six PSTs used the former and five used the latter. All the PSTs $(n = 11)$ accurately determined the order of operations to reflect the problem and established a correct mathematical expression for this problem. Brendan's written response, reproduced in **Figure *5.39***, is indicative of the way that the PSTs provided an expression in the form of $a \div b \times c$. Danny's written response is indicative of the way that the PSTs expressed the division as a fraction (see **Figure *5.40***).

Figure 5.39

Brendan's Response for Contextualised Problem 2a

Figure 5.40

Danny's Response for Contextualised Problem 2a



When the PSTs were asked to explain why they mathematised the problem the way they did, they based their explanations on the context of the problem. Brendan, for example, explained, "I need to divide before I multiply. I need to find the number of tyres produced in one day first only then I can find the number in 14 days." As he realised that the daily production rate needed to be determined first, he used of division at the start of the expression. After obtaining the daily production rate, he needed to get the production rate for 14 days and thus used of multiplication. Although the order of division and multiplication does not matter, the fact that he mathematised the problem in this order reflects his attempt to match the context to the written order of the corresponding operations. This suggests that Brendan was able to comprehend the underlying mathematical structure of the context and wrote the expression in a way that was consistent with the order of operations.

There was evidence that a PST used the correct order of operations but based on an incorrect explanation. Felix wrote the correct expression $6351 \div 3 \times 14$ to reflect Problem 2a, but his explanation was based on BODMAS. When asked why he used division first, he explained, "Following the BODMAS, division must be done first." This statement suggests that Felix determined the order of operations based on BODMAS and gave priority to division over multiplication. When further prompted why he recognised that the first operation was division, he simply referred back to

BODMAS. It is likely that Felix understood the context of the problem since he could use division to find the daily production rate at the start of the expression. However, his explanation was incorrect as he referred to BODMAS.

The second contextualised problem (Problem 2b) involved subtraction and division. This contextualised problem was posed as follows:

> *There are 532 parking spots on the first level of a multi-level parking lot and the rest of the parking spots are distributed equally on the other 8 levels. How many parking spots are there on the top level if there are total of 1532 parking spots?*

Problem 2b can be represented using the expression $(1532 - 532) \div 8$ or $\frac{1532-532}{8}$, in which five PSTs used the former and four used the latter. Audrey, Brendan, Casey, Danny, Eddy, Felix, Gavin, Howard, and Julie could determine the correct order of operations for this problem. Eddy's written response, reproduced in **Figure *5.41***, is indicative of the way that the PSTs provided an expression in the form $(a - b) \div c$. Danny's written response is indicative of the way that the PSTs expressed the expression in the form $\frac{a-b}{c}$ (see **Figure *5.42***).

Figure 5.41

Eddy's Response for Contextualised Problem 2b

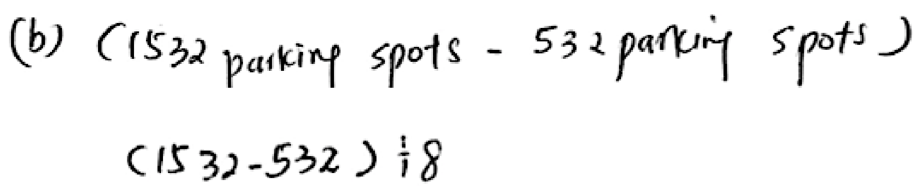
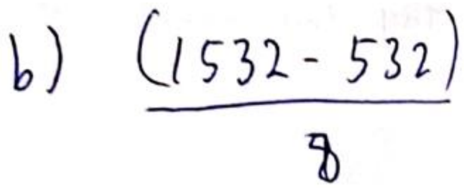


Figure 5.42

Danny's Response for Contextualised Problem 2b



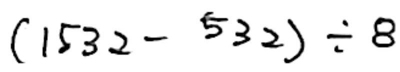
In the interviews, the PSTs were required to explain why they mathematised the problem the way they did. Audrey, Brendan, Casey, Danny, Eddy, Howard, and Julie referred to the context of the problem. Julie, for example, explained that:

Julie: First, we need to know how many parking spots are available on the other 8 levels. We cannot find the number of parking spots on the top level before finding the number of parking spots on all the 8 levels. To do so, we must use 1532 minus 532 because the total parking spots is 1532 and there are 532 parking spots on the first level. Then divide the answer by 8 to obtain the number of parking spots on the top level because they are divided equally on the 8 levels.

Julie's response shows that she made sense of the problem as having two steps. Based on her understanding about the context of the problem, she argued that the difference must happen first and used subtraction at the start of the expression to reflect this. As she realised that the number on the top level needed to be determined next, she used division. Julie appeared to grasp that there was a hierarchy between subtraction and division and thus used parentheses to highlight that priority was given to subtraction by stating, "I put brackets for 1532 minus 532 to show that I need to minus first." Her written response is reproduced in **Figure *5.43***.

Figure 5.43

Julie's Response for Contextualised Problem 2b

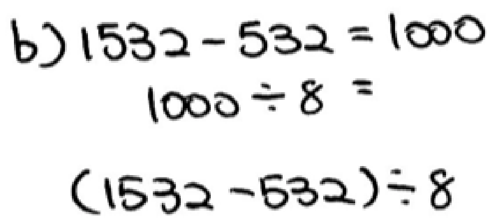


Julie's responses suggest that she was able to apply her knowledge of the use of parentheses to formulate an expression that reflected the context of the problem. Her use of knowledge of parentheses that reflects her knowledge of order of operations led her to use the convention to achieve the mathematisation.

In some cases (Felix, Gavin), BODMAS surfaced in the explanation. Felix, for example, stated that, "According to BODMAS, brackets must be simplified before division." When further prompted why he included parentheses enclosing the subtraction and computed subtraction first, he stated that, "Based on the rules, brackets B must be done first." Seemingly, Felix determined the order of operations based on BODMAS, but it is likely that he was able to apprehend the context of the problem (see **Figure *5.44***).

Figure 5.44

Felix's Response for Contextualised Problem 2b

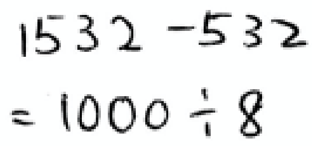


He probably realised that the difference between the total number of parking spots and the number of parking spots at the first level needed to be determined first and thus used subtraction at the start of the expression. However, he was unable to articulate how the order of operations provides a means of reflecting the context.

It is noted that Audrey wrote an invalid mathematical equation even though she could determine the correct order of operations for Problem 2b. Her written response, reproduced in **Figure *5.45***, shows that she first subtracted 532 from 1532, which equals 1000 then immediately, in the same equation denoted 1000 divided by 8.

Figure 5.45

Audrey's Response for Contextualised Problem 2b

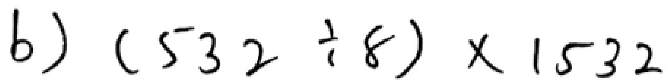


The response indicates that Audrey realised subtraction was the first operation reflected in the problem and division was the second. However, she appeared to lack an understanding about the concept of equality because $1532 - 532 \neq 1000 \div 8$. She stated that "I need to subtract with the first level first which is 1532 minus 532, so I get the answer 1000 and then I divided into other 8 levels." Audrey was correct in identifying the order of operations for this problem, but she did not express the order in a single valid mathematical expression. This is a common mistake in solving equations as noted in the study Knuth et al. (2006). A potential explanation for this might be that Audrey probably focused on getting the answer without paying attention in writing a valid mathematical expression. Another possible explanation for this might be that Audrey used the equal sign with the meaning of "results in", not equivalent.

In completing Problem 2b, Irene and Kevin wrote $532 \div 8 \times 1532$, which was incorrect. Irene's written response, reproduced in **Figure *5.46***, is indicative of the two PSTs used an incorrect expression.

Figure 5.46

Irene's Response for Contextualised Problem 2b



Irene explained that "Distributed equally on the other 8 levels means 532 divided by 8 and when we need to find the total of 1532 parking spots, we multiply it with 1532." The excerpt shows that Irene interpreted the phrase "distributed equally" correctly as the process of division but she appeared to have two problems. First, she failed to choose the correct operation (i.e., subtraction) to find the number of parking spots on the other 8 levels. Second, the order of the numbers she put in the expression was incorrect. Instead of dividing the difference between the total parking spots and the parking spots on the first level by 8, she divided the number of parking spots on the first level (i.e., 532) by 8. This result suggests that Irene did not completely understand the problem and struggled with comprehension of the context of the problem.

The third contextualised problem (Problem 2c) concerned three mathematical operations: addition, subtraction, and multiplication. This contextualised problem was posed as follows:

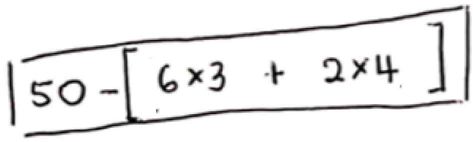
> *In a bookshop, paperback books cost $3 and hardback books cost $4 each. Alice buys six paperback and two hardback books. How much change will Alice receive from a $50 banknote?*

Problem 2c can be represented using the expressions $50 - (6 \times 3 + 2 \times 4)$ or $50 - 3 \times 6 - 4 \times 2$. Of the 11 PSTs, 10 (Audrey, Brendan, Casey, Danny, Eddy, Felix, Gavin, Howard, Julie, Kevin) could determine the correct order of operations to reflect this problem.

In the interviews, all the explanations provided by the PSTs is related to the context of the problem. Gavin's written response, reproduced in **Figure *5.47***, illustrates that the way the PSTs gave priority to finding the total cost of the books.

Figure 5.47

Gavin's Response for Contextualised Problem 2c



Gavin explained as follows:

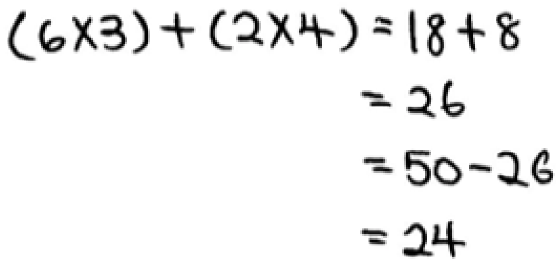
Gavin: To find the change, I must find the total cost first. Otherwise, there are different answers. For paperback is 6 times 3 and for hardback is 2 times 4. I must do multiplication first to see how much the paperback costed and how much the hardback costed. Then I must add these two costs to get the total cost. Lastly, I do minus to find the change.

The excerpt shows that Gavin presumed the problem as having multiple steps and recognised there was a hierarchy of operations to follow in order to arrive at the same answer. He realised that the total cost of the books needed to be determined first, which he manifested in the use of addition of two products. Finding the change from $50 is the next step. To reflect this, he used subtraction at the start of the expression and added parentheses to highlight that the total cost was given priority. He stated that, "I add the square brackets here at the addition to find the total cost before doing subtraction to find the change." Similar to the responses presented in Problem 2b, Gavin determined the order of operations based on the context of the problem and his knowledge of using parentheses led him to use parentheses to prioritise addition over subtraction to reflect the context of the problem.

It is noted that Felix wrote a series of invalid mathematical equations even though he could determine the correct order of operations for Problem 2c. His written response, reproduced in **Figure *5.48***, shows that he first determined the total cost of the books by summing the products of 6×3 and 4×2.

Figure 5.48

Felix's Response for Contextualised Problem 2c



Similar to Audrey's responses presented in Problem 2b, Felix used the correct order of operations to reflect the problem, but he displayed a common mistake in solving equations as claimed by Knuth et al. (2006). Although the instruction in the questionnaire clearly stated that the PSTs have to write a single mathematical expression to represent the problem, Felix did not express the order in a single valid mathematical expression. It is likely that he quickly combined steps to get the answer and used the equal sign as a way of listing each next step, not equivalent.

There was evidence that Irene was not able to use a correct order of operations to reflect Problem 2c. She gave an erroneous expression to this problem (see **Figure *5.49***).

Figure 5.49

Irene's Response for Contextualised Problem 2c

Irene was able to apply her knowledge of the use of parentheses to find the total cost of the books, but wrongly placed the 50 as the subtrahend instead of the minuend. She explained that, "Change means minus we need to use the answer just now to minus

50.” This explanation suggests that Irene correctly interpreted the word “change” as the need to use subtraction but the order of the numbers she put in the expression was incorrect. Instead of writing $50 - (6 \times 3 + 2 \times 4)$, she wrote $(6 \times 3) + (4 \times 2) - 50$. Apparently, Irene realised that the total cost of the books needed to be determined first and thus used addition of two products at the start of the expression. She was also aware that finding change of money was the next step that should be displayed in the use of subtraction, but she placed the subtrahend and the minuend incorrectly. This result suggests that Irene interpreted the problem literally causing her to match the context of the problem inaccurately.

The fourth contextualised problem (Problem 2d) involved operating on indices and multiplication. This contextualised problem was posed as follows:

> *Helen is enlarging a photo of 4 cm width on her tablet screen. The width of the photo is doubled each time she enlarges the photo. What is the width of the photo on her screen if she enlarges the photo five times?*

Problem 2d can be represented using the expression 4×2^5. Of the 11 PSTs, five (Brendan, Danny, Eddy, Gavin, Howard) recognised the order of operations and used a correct expression to represent the problem.

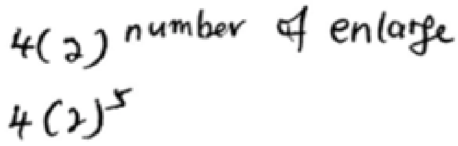
The PSTs were required to explain why they mathematised the problem the way they did. The explanation that emerged from the five PSTs who got the correct order of operations is related to the context of the problem. Eddy, for example, explained that:

Eddy: Enlarging a photo will double the width so times 2. Since Helen enlarges it five times, I write 2 to the power of 5 instead of 2. I have to know the degree of total enlargement before I can multiply it with 4.

Based on the context of the problem, Eddy correctly defined double as multiplying by 2. He considered finding the degree of enlargement (i.e., 2^5) must happen first before multiplying the width. Eddy's written response is reproduced in **Figure *5.50***.

Figure 5.50

Eddy's Response for Contextualised Problem 2d

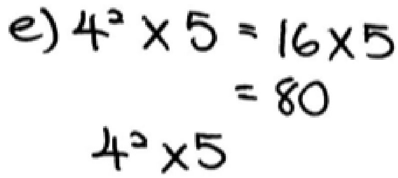


Eddy's responses suggest that he determined the order of operations based on the context of the problem. Realising the degree of enlargement needed to be determined first, and used an index to reflect this (i.e., 2^5). It is acknowledged that his understanding of indices also led him to write a correct expression to reflect the problem.

The analysis also shows that six PSTs gave erroneous responses to Problem 2d. Some of them misinterpreted the action of enlarging five times as to multiply by 5 and some appeared to lack an understanding of indices or area enlargement. **Figure *5.51*** shows Felix's erroneous response for the contextualised problem.

Figure 5.51

Felix's Erroneous Response for Contextualised Problem 2d



It is likely that these PSTs did not completely understand Problem 2d and struggled with comprehension of the context of the problem. It is acknowledged that this contextualised problem requires not only the knowledge of order of operations but also the knowledge of indices. The unsuccessful mathematisation of this contextualised problem might related to an inability to apply relevant mathematical knowledge to formulate an expression for the context.

To summarise this section, the PSTs' explanations were mainly based on the context of the problems. They considered each problem had one operation that must happen before another. The findings suggest that the use of contextualised problems potentially helps to see that there exists a hierarchy between operations. When presented with contextualised problems, the PSTs did not need to refer to the order conventions to determine the correct order of operations. The findings also indicate the danger of learning rules without understanding. The case in point is Felix's explanations. He could determine the order of operations to reflect the Problem 2b, but he used an incorrect explanation, which was based on BODMAS. In the long term, learning rules without understanding may lead to misinterpretations or misuse. Knowing how to evaluate expressions is something that anyone should know, but this is not enough for a PST. A PST needs to know why the answer makes sense in turn helps their students to be flexible and adaptive in applying the order of operations.

5.5 PRE-SERVICE TEACHERS' INTERPRETATION OF STUDENTS' WRITTEN WORK INVOLVING THE ORDER OF OPERATIONS

To explore how the PSTs interpret students' written work involving the order of operations, two hypothetical incorrect student responses were presented as Task 1 of the interview. In this task, the PSTs were asked to assess the written work and interpret the sense making underpinning the work. The results for each of the hypothetical student work are discussed in turn.

The first student work reflected the error of performing multiplication before division (see **Figure *5.52***).

Figure 5.52

First Hypothetical Student Work of Task 1

$$\begin{aligned} 18 \div 3 \times 2 &= \frac{18}{3 \times 2} \\ &= \frac{18}{6} \\ &= 3 \end{aligned}$$

i) Assess Michelle's responses.
ii) Why do you think she evaluated the expression that way?
What misunderstandings/misconceptions is she likely to have?

This error was due to the student interpreting $18 \div 3 \times 2$ as $18 \div (3 \times 2)$ and had given priority to multiplication. Of the 11 PSTs, six (Audrey, Brenda, Eddy, Felix, Gavin, Kevin) were able to identify the error, whereas five (Casey, Danny, Howard, Irene, Julie) were not able to determine the error. Audrey's response, reproduced in **Figure *5.53***, illustrates the way that the six PSTs determined the student's error.

Figure 5.53

The Correct Solution for $18 \div 3 \times 2$ *– Audrey's Response*

$$\begin{aligned} 18 \div 3 \times 2 &= \frac{18}{3} \times 2 \\ &= 6 \times 2 \\ &= 12 \end{aligned}$$

The PSTs' responses were analysed to identify how they interpreted the written work. Half of the PSTs, who determined the error (Brendan, Gavin, Kevin), interpreted the student's work mathematically. Using a mathematical approach to interpret students' written work, as suggested by Baldinger (2020), involves utilising PSTs' understanding of the mathematical content to draw conclusions or to evaluate the methods students use to arrive at solutions. Gavin, for example, stated that, "This question should be solved from left to right, or you can use any order if you convert division to multiplication because division is multiplication by the opposite. The student might not know this, so she simply multiplied first." This excerpt implies that Gavin interpreted the student's work based on his understanding about division means multiplicative inverse and he related this to the order of operations. He explained that the error was due to the student disregarding the relationship between multiplication and division.

There was evidence that another half of the PSTs, who determined the error (Audrey, Eddy, Felix), interpreted the written work pedagogically. Interpreting students' written work through a pedagogical approach involves utilising PSTs' understanding of common student approaches and general methods of teaching a mathematical concept (Baldinger, 2020). Eddy brought in the idea of common errors to support his interpretation. He explained that, "There are usually two types of errors. The first type always prioritises multiplication over division, and the second type always prioritises division over multiplication. This student's error falls in the first type." He further interpreted the error was due to limited time of instruction. He explained that, "Given a few mathematics lessons a week, teachers need to cover many topics. Less time is allocated to train students in solving such questions, so students easily get confused and make errors." Although Eddy did not explain from the perspective of students' thinking, his used a context-based interpretation that is in terms of instructional time. Audrey and Felix, on the other hand, interpreted the error as a result of the student not remembering BODMAS. Felix, for example, stated that, "She might not remember BODMAS which is usually used by the teacher to teach the order of operations." The responses given by these three PSTs showed that they reasoned pedagogically when interpreting the student's work. To support their interpretations, they drew on their knowledge about what errors students in general might make and how the order of operations might commonly be taught.

The analysis shows that four PSTs, who were not able to determine the error (Casey, Danny, Howard, Irene), interpreted the student's work through self-comparison, that is, they compared the student's solution to their own solution. This involves identifying both similarities and differences between the solution proposed by the PSTs and that of the students (Baldinger, 2020). For example, Howard stated that, "I think she did it correctly like what I did in the previous questions. I mean she can do multiplication first only then do division just like what I did." This excerpt indicates that Howard compared what the student did and what he himself had used in his evaluation. However, his perception that multiplication must be computed before division was incorrect. He made the same error as the student did.

The data also show that Julie was not able to identify the error and interpreted the student's work pedagogically. She stated that, "It is okay to convert the question to a fraction like this because normally after getting a fraction we teach students to simplify the denominator first." This excerpt implies that Julie relied on her knowledge about what students might generally do when dealing with fractions. However, she did not notice that the student incorrectly transferred the expression into a fraction form that had a different meaning. She made the same error as the student.

The second hypothetical student work presented another order of operations error in which the student calculated division before addition for an expression that was in a fraction form (see **Figure *5.54***).

Figure 5.54

Second Hypothetical Student Work of Task 1

$$\frac{50+28}{5+7} = \frac{10+4}{1+1}$$
$$= \frac{14}{2}$$
$$= 7$$

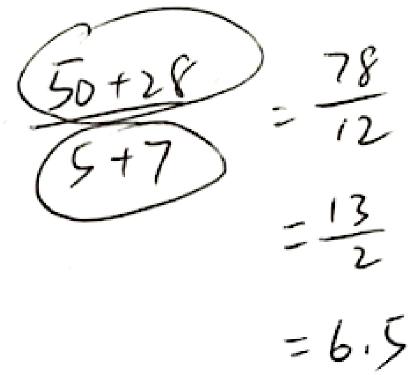
i) Assess Michelle's responses.

ii) Why do you think she evaluated the expression that way?
What misunderstandings/misconceptions is she likely to have?

The student cancelled common factors even though the numerator and denominator were both sums. All the PSTs identified the error, but they used two different approaches to interpret the written work. Julie's response, reproduced in **Figure *5.55***, illustrates the way that the PSTs determined the student's error.

Figure 5.55

The Correct Solution for $\frac{50+28}{5+7}$ *– Julie's Response*



First, there was evidence that 10 PSTs interpreted the student's work mathematically. Kevin, for example, stated that:

Kevin: Let's check. The student simplified 50 and 5 so she got 10 over 1, simplified 28 and 7 so she got 4 over 1. Then she summed up to get 14 over 2, which seems correct, but she is wrong. She should have summed up 50 and 28 to get 78, and summed up 5 and 7 to get 12 first before doing division.

Apparently, Kevin tried out the student's solution and checked if the computations make sense. Although the computations were correct, Kevin reasoned that the student used an incorrect order of operations. He further interpreted the student's error was due to the failure in seeing the fraction bar as a grouping symbol. He explained that, "She got this question wrong maybe because she immediately interpreted the fraction bar as division, but the fraction bar could also imply grouping. It grouped 50 and 28 as the numerator and grouped 5 and 7 as the denominator.” Kevin’s interpretation was mathematical in nature as he drew on his understanding about a fraction bar, that is, viewing the fraction bar as having two functions: a grouping symbol and the operation of division. He explained that the student might have viewed the fraction bar only as division thus causing the student to cancel common factors even though the numerator and denominator were both sums.

Second, there was evidence that Julie interpreted the student’s work pedagogically. She stated that, “Since teachers always ask students to convert a fraction to its simplest form, this student might think that she had to do the same thing, so she simplified the numerator with the denominator when solving this question.” This excerpt suggests that Julie drew on her knowledge about how fraction operations might generally be taught to support her interpretation. In other words, Julie interpreted the student’s work based on the common teaching practices of fractions. In addition to this, Julie also made a context-based interpretation. She further explained that, “The error may because of teachers did not have enough time to use various teaching materials to teach the topic.” Similar to Eddy’s response to the first hypothetical student work, Julie used a context-based interpretation that is in terms of instructional time to reason the second hypothetical student work.

To summarise this section, analysis of the PSTs’ responses revealed three approaches in interpreting the written work: mathematical, pedagogical, and self-comparison. **Table 5.6** shows the approaches the PSTs used to interpret the written work. The interpretation was largely mathematically as the PSTs primarily engaged in mathematical reasonings. Interpretations that were mathematical in nature tended to lead the PSTs to determine the error. Although some PSTs were not interpreted the written work mathematically, they were able to see foundations of the errors, such as the teaching practices of fraction operations, rather than merely focused on order of operations.

Table 5.6

Approaches Used in Interpreting Written Work

Student work		Approaches used to interpret student's work
First hypothetical student work	Able to determine the error	Mathematical Pedagogical
	Unable to determine the error	Self-comparison Pedagogical
Second hypothetical student work	Able to determine the error	Mathematical Pedagogical

5.6 PRE-SERVICE TEACHERS' PEDAGOGICAL KNOWLEDGE OF THE ORDER OF OPERATIONS

This section presents the results of the interview and the PSTs' lesson plans. Task 2 of the interview (Section 4.5.2, p. 68) was designed to determine the approach the PSTs think is best used to teach the order of operations. In this task, three classroom practices were presented to the PSTs, and they were required to choose and justify the classroom practice they think is the most effective. They were also asked to suggest ways to prevent students from making errors and misinterpreting the order of operations. The PSTs' lesson plans, which were developed before the interviews, serve as another data source to examine how they plan to approach the topic.

In general, the PSTs demonstrated different ways to approach the order of operations. **Table 5.7** shows the teaching approaches of the PSTs. Each of the approaches are explained further in the following sections.

Table 5.7

Teaching Approaches of PSTs

Evidence in lesson plan	Evidence from Task 2		
	Transmission	Connectionist	Discovery
Transmission	Felix Kevin	Audrey Brendan Casey Irene	Danny Eddy Gavin Howard
Connectionist			Julie
Discovery			

5.6.1 Transmission Approach

The responses provided by Felix and Kevin were consistent with a transmission approach. As defined in Section 2.6.2 (p. 41), the transmission approach involves a teacher's explanation followed by giving routine exercises. This approach focused primarily on rote learning and drilling in order to develop students' competency in doing mathematics.

When assessing the classroom practices, Felix and Kevin chose Tracy's practice (Transmission classroom) as the most effective way to teach the order of operations. For example, Kevin explained that:

Kevin: Tracy is good because she gave a lot of explanations to introduce the rules. She used BIDMAS is a very good way to remember the rules. Moreover, a lot of questions were given to students to try so that they can see the pattern of calculations clearly. By using BIDMAS, they can solve every question step by step easily.

Kevin's excerpt about an introductory explanation of the order of operations suggests that he grasped the transmission approach. Use of BIDMAS as a way of remembering the order of operations suggests rote memorisation is important to Kevin. He viewed BIDMAS as a tool to evaluate expressions gradually from one step to another. For Kevin, use of sets of questions suggests drilling is essential in the teaching of the order of operations.

When asked about ways to prevent students from making errors and misinterpreting the order of operations, the PSTs' responses imply a transmission approach. For example, Felix explained that, "More practices can help students to imitate the same working accurately, so we have to instruct them to write down every step on paper clearly." For Felix, drilling using sets of questions could train students to use procedures. Asking students to list every step of calculations suggests Felix viewed explicit instruction as an important element to prevent students' errors and misinterpretations.

An analysis of the PSTs' lesson plans shows that their lessons were consistent with a transmission approach. The planned activities primarily focused on giving verbal explanations and performing procedures based on the order of operations. Quotations from the lesson plans that provide insight into the characteristics of transmission approach are presented in **Table 5.8**.

Table 5.8

Quotations From PSTs' Lesson Plans That Show Transmission Characteristics

Quotations from lesson plan	PST's lesson plan
"Explains to student what the order of operation is. Students read aloud the order of operations and repeat reading for 10 times to remember the rules." "In groups, students compare their final answers and submit their work to the teacher within 15 minutes."	Felix
"Based on the textbook, teacher explains the order of operations again before doing exercises." "Have a Q&A sessions in which one student reads out a question from the worksheet and another student gives the answer. This continues in the same way for all the students."	Kevin

As presented in **Table 5.8**, explaining the order of operations explicitly suggests both Felix and Kevin grasped the transmission approach. In addition to verbal explanations, repeat reading aloud the order of operations as a way of remembering suggests Felix focused on rote memorisation in his planning. Setting a time limit for students to submit their work suggests Felix expected his students to have a rapid recall of the procedures and to be able to simplify expressions quickly. The quotations from **Table 5.8** also indicate limited interactions between the PSTs and students when performing procedures of the order of operations. Although the planned activities were seemingly carried out in groups or the whole class, the students had in fact applied the order of operations to evaluate expressions individually before reporting their answers to other students.

The lesson plans also indicate that the PSTs did not make an explicit link between the order of operations and the properties of operations. They simply included an instruction such as, "Asks students to look at the definition of the laws of arithmetic operations," and "Explains the laws of arithmetic operations to students." Overall, the analysis revealed that Felix and Kevin planned for a transmission approach because they relied on the teacher's verbal explanations, focused on students' rote memorisation, involved limited classroom interactions, and expected rapid recall of the procedures among students.

5.6.2 Combination of Transmission and Connectionist Approaches

The responses provided by Audrey, Brendan, Casey, and Irene were consistent with a combination of transmission and connectionist approaches. In addition to the transmission approach as discussed in Section 5.6.1 (p. 151), these PSTs also grasped a connectionist approach. As defined in Section 2.6.2 (p. 41), the connectionist approach involves linking different mathematical contents, emphasising students' explanation and interactions in teaching, recognising students' errors, and refining students' methods. Since the characteristics of the transmission approach have been presented in the previous section, the following paragraphs presents the characteristics of the connectionist approach.

When assessing the classroom practices, these PSTs chose Connie's practice (Connectionist classroom) as the most effective way to teach the order of operations. For example, Brendan explained that, "Connie asked students to explain their methods used in solving the problem, which is good to train reasoning skill. By doing so, Connie will know if the students have misunderstanding." Requesting students to justify their work suggests Brendan grasped the connectionist approach. This excerpt also implies that making sense of students' thinking is important to Brendan.

When asked about ways to prevent students from making errors and misinterpreting the order of operations, the PSTs' responses imply a connectionist approach. For example, Brendan explained that, "Let students know that $(a + b) + c = a + (b + c)$ and let them realise that $-a = +(-a)$. This can help them to know that calculations for addition and subtraction can be done from left to right. Same thing goes to multiplication and division." The excerpt implies that making mathematical connections is important to Brendan as making such connections may help students to make sense of the left-to-right order.

An analysis of the PSTs' lesson plans shows that their lessons were consistent with a connectionist approach. Quotations from the lesson plans that provide insight into the characteristics of connectionist approach are presented in **Table 5.9**. Audrey's activity suggests that making students' errors explicit and refining the students' methods are important to her. Irene planned for a class discussion and pair works indicate she emphasised student interactions with others. For Irene, student learning is based on negotiating meanings with the teacher and students. Brendan attempted to relate the order of operations to the properties of arithmetic operations using the proposed expression in **Table 5.9**. This suggests that helping students to make mathematical connections is important to Brendan. However, it is unclear how Brendan established a link between these two plans.

Table 5.9

Quotations From PSTs' Lesson Plans That Show Connectionist Characteristics

Quotations from lesson plan	PST's lesson plan
"Students who have incorrect steps will be grouped together and the teacher will point out where their mistakes are. Using another similar question, the teacher shows to students the correct method and students will need to solve the question again."	Audrey
"The teacher discusses with the whole class what are the possible ways to evaluate expressions involving addition and subtraction. Then, in pairs, students discuss another question involving multiplication and division."	Irene
"In groups, students calculate $(\mathbf{3}+\mathbf{8})+\mathbf{4}$ and $\mathbf{3}+(\mathbf{8}+\mathbf{4})$ based on the order of operations then compare the steps and the answers. Introduce to students the associative law."	Brendan

5.6.3 Combination of Transmission and Discovery Approaches

The responses provided by Danny, Eddy, Gavin, and Howard were consistent with a combination of transmission and discovery approaches. In addition to the transmission approach as discussed in Section 5.6.1 (p. 151) these PSTs also grasped a discovery approach. As defined in Section 2.6.2 (p. 41), the discovery approach involves providing opportunities for students to discover methods for themselves and recognising students vary in learning ability. This approach focused primarily on students' preparation and action on objects. Since the characteristics of the transmission approach have been presented in the previous section, the following paragraphs presents the characteristics of the discovery approach.

When assessing the classroom practices, these PSTs chose Dickson's practice (Discovery classroom) as the most effective way to teach the order of operations. For example, Danny explained that:

> Danny: Dickson is the best because he didn't care about the answers, he wanted his students to evaluate the expressions by any method. What is important here is that the students learn through working out the expressions themselves and identify the pattern to solve the questions correctly.

Getting students to discover the correct method suggests Danny grasped the discovery approach. He valued the process of creation of methods rather than the final answers. For Danny, student understanding is developed from finding their own ways of calculations.

An analysis of the PSTs' lesson plans shows that their lessons were consistent with a discovery approach. Quotations from the lesson plans that provide insight into the characteristics of discovery approach are presented in **Table 5.10**.

Table 5.10

Quotations From PSTs' Lesson Plans That Show Discovery Characteristics

Quotations from lesson plan	PST's lesson plan
"Ask students to read page 10-12 before the class next week then make short note for themselves after reading."	Gavin
"Give 20 questions and ask students to solve the questions using any method they think is correct. After completing the questions, answers are provided for students to check. Students discover the correct method and write down the correct order of operations. The teacher summarises the correct order of operations by writing the correct order on the board."	Danny
"Dividing students into three groups – good, average, low attaining. Group 1 - Good students solve 25 questions. Group 2 - Average students solve 20 questions. Group 3 - Low attaining students solve 10 questions with the teacher's guidance, small cubes will be given for students to count."	Howard

Gavin's activity that asking students to read and prepare before a lesson suggests students' preparation is important for Gavin. He expected students to be ready prior to learning the order of operations. Danny's activity that giving an opportunity for students to discover the order of operations suggests he perceived student-led exploration and reflection as a crucial element in learning the topic. For Danny, the pace of learning is determined by students, not the teacher. Howard's activity shows two characteristics of the discovery approach. First, dividing students into different groups of ability suggests Howard recognised students vary in the rate at which they learn the order of operations. Second, providing small cubes to manipulate suggests Howard viewed the action on objects is important to learn the order of operations.

5.6.4 Combination of Discovery and Connectionist Approaches

The responses provided by Julie were consistent with a combination of connectionist and discovery approaches. When assessing the classroom practices, Julie chose Dickson's practice (Discovery classroom) as the most effective way to teach the order of operations. She explained that, "The task given by Dickson is good to allow students to think of how to solve the expressions. Students will understand and not forget so easily the correct calculation if they find out the correct way themselves." Allowing students to find ways of calculations for themselves suggests that students' own strategies are important to Julie. For Julie, student understanding is based on working the method out themselves.

When asked about ways to prevent students from making errors and misinterpreting the order of operations, Julie's response implies a discovery approach. She stated that:

Julie: I think the problem might be the students do not remember multiplication table, and do not know how to divide, so they cannot solve questions with multiple operations. I think I will request the students to learn the multiplication table again before learning the order of operations.

Revisiting the multiplication table suggests that Julie viewed students' errors and misinterpretations of the order of operations as the result of students not prepared to learn the order of operations. She wanted to make sure students are ready prior to evaluating expressions with multiple operations.

An analysis of Julie's lesson plans shows that her lessons were consistent with a connectionist approach. Quotations from the lesson plans that provide insight into the characteristics of connectionist approach are presented in **Table 5.11**.

Table 5.11

Two Activities From Julie's Lesson Plans That Show Connectionist Characteristics

Quotations from lesson plan
"Write the number 5 on the board and ask students to make as many valid equations as possible based on their current knowledge. For example, $\mathbf{25 \div 5}$. Discuss the use of brackets in the order of operations with students then require students to try again writing as many valid equations as possible using brackets. For example, $(\mathbf{4+1}) \times \mathbf{5}$. The teacher and students summarise the order of operations based on the activity."
"Number talk activity. Prepare four strips with different questions written on it. Four students will be called randomly and they will need to choose a strip and talk about the question they have got. 1. What are the order of operations? 2. Why do you think the multiplication and division are completed before the addition and subtraction? 3. Why do you think the exponents need to be evaluated before multiplication and division? 4. How are the order of operations and the distributive law connected?"

The first activity as presented in **Table 5.11** shows three characteristics of the connectionist approach. Asking students to write expressions based on their understanding about the four basic arithmetic operations suggests Julie viewed students' previous knowledge as essential in learning the order of operations. Conducting a discussion among the teacher and students suggests classroom interactions are important to Julie. Requiring students to rewrite expressions suggests Julie perceived students learn the order of operations through overcoming challenges. For the number talk activity as presented in **Table 5.11**, asking students to explain the order of operations and the properties of operations suggests making mathematical connections is important to Julie. Using Why and How questions indicates that Julie emphasised students' reasoning in her planning to teach the order of operations. Overall, the analysis revealed that Julie had discovery beliefs about how best to teach the order of operations, but in planning she employed the connectionist approach.

To summarise this section, most PSTs in this study had characteristics of more than one approach. Of the 11 PSTs, two PSTs had the characteristics of a transmission approach. It is suggested that these PSTs may consider focusing students' sense making through giving opportunities for students to explain their work or discover their own method to learn the order of operations. Other nine PSTs used various methods to approach the order of operations. It is likely that these PSTs may have shifted their attention from performing routine procedures to building sophisticated thinking about the order of operations.

5.7 INTERPRETING THE EMPIRICAL EVIDENCE IN TERMS OF THE CONCEPTUAL FRAMEWORK

The initial framework was based on existing teacher frameworks from the literature (e.g., Askew et al., 1997; Ball et al., 2008; Chick et al., 2006; Tatto et al., 2008). However, based on the collected data, the participants did not necessarily make sense of the order of operations based on a correct and useful connections. The participants also did not necessarily interpret students' written work based on what the students found easy or difficult. Therefore, the conceptual framework was refined to address the mathematical pedagogical content knowledge required for the order of operations (see **Figure *3.1***, p. 51). This study, thus, conceptualises mathematical pedagogical content knowledge as knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches.

The collected data showed that the participants' knowledge of content was based on different mathematical connections. The connections made were either useful or problematic when the participants made sense of the order of operations. For example, Julie made a relevant and useful connection by linking the order of operations to the concept of set. Based on this connection, she made sense of the left-to-right order correctly. There was evidence that the participants' knowledge of content was based on problematic connections. For example, Howard provided two answers (4 and 1) to expression $10 \div 5 \times 2$ based on his understanding that a quadratic equation has two roots. The connection, however, is problematic. On another occasion, Julie referred to set notations and set operations when making sense of the left-to-right order. This connection was invalid in justifying why the left-to-right order worked. Therefore, the developed conceptual framework is used to address variations of PSTs' explanations.

The data also indicated that the participants' knowledge of students' sense making was mostly based on a mathematical approach when interpreting students' written work. For example, Gavin analysed the student's written work based on his understanding that division means multiplicative inverse and he related this to the order of operations. Although a self-comparison approach was used occasionally, this approach was problematic to the participants because they based their interpretations on their own incorrect solutions. In other words, they made the same errors as the student. In this respect, the conceptual framework formulated in this study is used to understand how PSTs interpret students' sense making.

In terms of knowledge of teaching approaches, the data showed that the participants used primarily a transmission method when approaching the order of operations. For the participants, verbal explanations and mnemonics were considered effective to teach the order of operations. They also believed that the topic was best learned through the paper and pencil method. There was evidence that the participants used a combination of different teaching approaches. For example, Audrey utilised a combination of transmission and connectionist approaches. In addition to a collection of routines, she planned for extensive dialogues to challenge students' current levels of thinking. The conceptual framework, thus, addresses different teaching approaches used for teaching the order of operations. All the aforementioned findings indicate the sophisticated nature of mathematical pedagogical content knowledge of PSTs.

Chapter 6: Discussion

6.1 CHAPTER INTRODUCTION

This chapter discusses findings that emerged from the analysis of the questionnaire, interview transcripts, and lesson plans. The findings are combined to evaluate the PSTs' mathematical pedagogical content knowledge of the order of operations. Throughout the discussion, research implications and suggestions for future research are signposted.

As discussed in Chapter 3 (p. 47), the conceptual framework of Mathematical Pedagogical Content Knowledge was developed to describe the nature and extent of the knowledge needed for teaching the order of operations. This model consists of three main elements that are about interpreting the content for teaching, making sense of students' difficulties, and planning to teach the content. Specifically, the model combines knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches. This chapter discusses these three knowledge components in turn.

In general, Section 6.2 discusses the analysis of knowledge of content, and in so doing, provides answers to the following research questions:

How do pre-service secondary mathematics teachers apply the order of operations to evaluate mathematical expressions?

How do pre-service secondary mathematics teachers interpret the connections between the order of operations and the properties of operations?

How do pre-service secondary mathematics teachers determine the order of operations of contextualised problems?

The PSTs' knowledge of students' sense making was analysed in terms of students' errors and misinterpretations of the order of operations. Section 6.3 discusses the analysis of knowledge of students' sense making and provides answers to the following research question:

How do pre-service secondary mathematics teachers interpret students' written work involving the order of operations?

Knowledge of teaching approaches was analysed in terms of the PSTs' teaching orientations and ways of eliminating errors in the order of operations. Section 6.4 presents a discussion about this analysis, and provides answers to the following research question:

How would pre-service secondary mathematics teachers plan to teach the order of operations?

6.2 KNOWLEDGE OF CONTENT

The first goal of the present study was to explore knowledge of content in relation to the order of operations possessed by the PSTs. Knowledge of content concerns the body of knowledge that PSTs must master to be effective in teaching. In this section, the findings about the PSTs' knowledge of content are presented in terms of how they apply the order of operations to evaluate mathematical expressions, make relevant connections between the order of operations and the properties of operations, and determine the order of operations of contextualised problems. Each of these elements of knowledge of content will be discussed in turn.

6.2.1 Interpretation of Mathematical Expressions

The order of operations is needed to simplify mathematical expressions and this section discusses how the PSTs used the ordering for this purpose. Specifically, this section discusses the results presented in Section 5.2 (p. 93) in order to answer the first research question:

How do pre-service secondary mathematics teachers apply the order of operations to evaluate mathematical expressions?

The findings suggest that the PSTs applied the left-to-right order based on associative and inverse properties. Although this was not obvious when the PSTs evaluated the expression with multiplication and division, it was apparent when they evaluated expressions involving addition and subtraction. The PSTs made sense of the rule through interpreting subtraction as additive inverse and addition as associative. Building on these interpretations, the PSTs exhibited flexibility in evaluating expressions involving addition and subtraction. They were able to perform calculations using any orders without affecting the final answer. Having this flexibility is imperative for PSTs so that they can assist their students to make sense of the rule. This finding contradicts Zazkis's (2018) argument, namely, that knowledge about the properties of operations was not used by the PSTs to interpret the order of operations. However, in the present study, the PSTs' knowledge about the associative and inverse properties indeed enabled them to make sense of why expressions with addition and subtraction and expressions with multiplication and division must be evaluated from left to right. As Zazkis (2018) investigated the interpretations of the left-to-right order for multiplication and division, this study extends her research by providing empirical evidence on how PSTs make sense of all the rules of the order of operations.

However, this flexibility was not obvious when the PSTs evaluated expressions with multiplication and division. If they can evaluate from left to right for expressions with addition and subtraction based on associativity of addition and additive inverse, it is likely that they also can make sense of the order of computations based on associativity of multiplication and multiplicative inverse. The finding, however, shows that the PSTs exhibited less flexibility in evaluating expressions with multiplication and division. This is probably because additive inverse is more common for the PSTs as a negative number, but multiplicative inverse is uncommon for them as a unit fraction.

There was evidence that the PSTs evaluated expressions based on the acronym used to recall the order of operations. Generally, they displayed fluency in recalling the order of operations and executing the procedures. Using the acronym, they were able to prioritise parentheses and exponents before all four basic operations. Some of them, however, misinterpreted the order of multiplication and division in relation to the order of letters presented in the acronym. They incorrectly performed division before multiplication because they misinterpreted the acronym BODMAS based on the order in which the letters DM were presented. This finding is different from Zazkis and Rouleau's (2018) study in which their Canadian PSTs misinterpreted the rule based on another acronym BEDMAS. In fact, there are research studies that argue multiplication before division using PEMDAS or BOMDAS. In one such study conducted by Glidden (2008), they found that their teacher participants prioritised multiplication over division, but they did not provide direct evidence for the root cause of the misinterpretation. The finding of the present study thus extends Glidden's (2008) study on the source of the misinterpretation. It also revealed the danger of using acronyms in the order of operations, regardless of what acronyms are used.

The analysis shows that the PSTs used the order of operations based on a hierarchical triangle similar to the representation suggested in past research (Ameis, 2011; Bay-Williams & Martinie, 2015). This visual representation scaffolds the PSTs to draw conclusions as the hierarchical triangle clearly shows the operations that are of the same priority (i.e., addition and subtraction, multiplication and division). It is acknowledged that visual representations can assist the memorisation and application of the order of operations; however, the reasons why the rules work cannot be underemphasised. Knowing the underlying reasons of the rules may develop a deeper understanding not only about the order of operations itself but also about the concepts that are undervalued in mathematics, such as associativity (Dupree, 2016).

When evaluating the expression with stacked exponents, some of the PSTs referred to the Power Rule of Exponents, which is an incorrect interpretation. When there is no indication of grouping for stacked exponents, they presumed 2^{3^2} means $(2^3)^2$ and applied the Power Rule inaccurately. This misinterpretation may be due to the ordering of stacked exponents is not given attention in relation to the order of operations, thus there appears to be a need to establish clarity and refinement on the order of operations (Lee & Messner, 2000). Although stacked exponents were not common in middle-school mathematics and may appear odd for the PSTs, the difficulty in evaluating stacked exponents should not be overlooked due to its real-life applications. For example, stacked exponents are commonly used in expressing population growth, such as e^{x^2}.

There is very little research about how PSTs interpret the rule used to evaluate exponents. Although Glidden (2008) revealed that 80% of the 381 PSTs in his sample made errors in simplifying expressions that contained indices, his study lacked the data that could indicate how the PSTs interpreted the order of operations that contributed to these errors. The findings of the present study represent an advance in the literature about the ways in which the stacked exponents are evaluated. However, not all the designed expressions could capture the kind of data that can show the PSTs' interpretation about the conventional rules. It is acknowledged that the present research design was limited in terms of the interrogation of exponent rules. The relationship between exponent rules and the order of operations is, therefore, worth examining further.

With regard to parenthesis, the PSTs appeared to not grasp why they were given priority. Although some compared the ordering with the distributive property and referred to other representations of the order of operations, in most cases the PSTs offered no reason and simply stated "I don't know." This may be because of the use of parentheses in relation to the order of operations is an arbitrary convention and the PSTs' knowledge about the historical antecedents of the order is superficial. Despite not knowing why parentheses are given priority, all the PSTs correctly performed operations within parentheses first. This finding is different from Dupree (2016) where she argued that parentheses may cause confusion with respect to the order of computations. In the present study, however, the PSTs' interpretations of the order of operations do not reflect any misinterpretations of the conventions.

The findings also suggest that the PSTs emphasised brackets as a strategy to facilitate the evaluation of expressions. This approach is "superfluous" (Gunnarsson & Karlsson, 2014, p. 53) because the value of an expression remains even when emphasising brackets in the expression is removed. As argued by Gunnarsson et al. (2016), the PSTs' use of this approach might be a result of perceiving emphasising brackets as a replacement for the order of operations. Therefore, it is worth investigating further the use of this approach in relation to the order of operations.

In some cases, the PSTs used examples when applying the order of operations to evaluate expressions. Take for instance, Julie used an example of area to explain why priority was given to exponentiation over subtraction. Use of examples is crucial because it reflects the PST's presentation of ideas regarding the topic (Rowland, 2013).

In other cases, the PSTs made analogical connections that were either relevant or irrelevant by referring to other mathematical concepts. Rodríguez-Nieto et al. (2022) describes an analogical connection as a connection that is made when a familiar situation is linked with a concept. However, this description lacks precision. In essence, reasoning with analogies refers to seeing inherent corresponding relationships between two situations or ideas (English, 2004; English & Sharry, 1996). Julie made a relevant connection as she linked the order of operations to the concept of set and made sense of the left-to-right order correctly. On the other hand, Irene made an irrelevant connection as she connected the order of operations with quadratic equations and interpreted that order of computations does not matter. Despite being able to connect the order of operations with other mathematical concepts, these connections were unable to explain why the left-to-right order worked. Therefore, making mathematical connections can result in meaningful learning of mathematics only when the connections made are appropriate.

The finding that Julie referred to set notations and set operations to make sense of the left-to-right order is important because it shows that learning is not necessarily from simple to complex, it may be that new learning impacts on prior knowledge. This finding is similar to past studies that examined the effect of new learning on prior knowledge (Cook, 2003; Hohensee, 2014; Tall, 2013). For example, Hohensee (2014) found that new learning of quadratic functions affected either positively or negatively on students' conceptual understanding of previously learned linear functions. However, these studies use different notions to describe the situation. Cook (2003) and Hohensee (2014) for example, use the notion *backward transfer* whereas Tall (2013) uses the notion *met-after*. Tall's (2013) met-after explains the effect of an experience met later in learning development that profoundly changes the way in which earlier ideas are considered. In this study, Julie's responses imply the learning of set, which was learned later, impacted her sense making of the order of operations, which was learned earlier in her mathematics learning development.

The findings on how PSTs simplify expressions create awareness of the need of the order of operations. For example, Casey was uncertain whether to evaluate Expression 1a without a specific order or calculate it from left to right. Although she decided to use the left-to-right order after consideration, her explanation was based on the fact that it was impossible to have two answers for one expression. On the other hand, the explanations from Howard and Irene that were based on quadratic equation and quadratic formula also pointed to the need of consistency. To achieve consistency in mathematics, the order of operations is necessary (Papadopoulos, 2015).

Understand how PSTs evaluate mathematical expressions and justify their ways of computations represent a contribution to the existing literature about the order of operations. The present study also highlighted misinterpretations of the order of operations that PSTs had. Further research might examine how the misinterpretations can be removed. Nevertheless, the evidence presented in the present study signals that PSTs exhibited flexibility in evaluating mathematical expressions when they recognised the connections between the order of operations and relevant mathematical concepts, such as associativity and inverses. As claimed by Van de Walle and Folk (2004), understanding is developed when quality and quantity of connections were made between a new idea and existing knowledge. The implication for PSTs to reason the seemingly arbitrary conventions is thus to ensure they teach mathematics with understanding, not focusing solely on procedural proficiency.

6.2.2 Connections Between the Order of Operations and the Properties of Operations

Teaching mathematics requires helping students to make connections between different mathematical ideas (Bansilal, 2014; Toh & Choy, 2021). Therefore, it is crucial that PSTs can recognise these connections before making them explicit and comprehensible to students. In this study, Item 3 of the questionnaire asked PSTs to interpret the connections between the order of operations and the properties of operations (see in Section 4.5.1, p. 63). This section discusses the results of Item 3 and contains answers to the second research question:

How do pre-service secondary mathematics teachers interpret the connections between the order of operations and the properties of operations?

The mathematical connections discussed in this study are from linking the order of operations to the properties of operations. Before discussing these relationships any further, it is worth considering the operations of addition and subtraction in a calculation. Typically, the correct order of computations for addition and subtraction is simply going left to right. Learners often presume this as a rigid order and if this order is not followed, it may lead to a different answer (Zazkis & Marmur, 2018). In this sense, $(4 - 3) + 2 \neq 4 - (3 + 2)$. However, this is not true for an expression that is in the form of $a + b - c$ (e.g., $4 + 3 - 2$). Either following the left-to-right order or performing out of order will lead to the same answer. In this study, the PSTs interpreted these orders in different ways.

The results show that the PSTs' explanations centred mainly around the following:

- Both orders work because subtraction is additive inverse and addition is associative.
- Both orders work because the answers are the same.
- Addition must be performed before subtraction because BODMAS indicates a rigid order in which addition is performed before subtraction.

The first explanation that PSTs used was that subtraction is additive inverse and addition is associative, leading to the conclusion that the order of computations does not matter for expressions involving addition and subtraction. This explanation is an indication that the PSTs made connections between the left-to-right order and the associative and inverse properties. They generalised the associative property and showed that the rule was constructed upon the properties of operations. This connection type is termed as a deviation by Eli et al. (2011) and part-whole connection by Rodríguez-Nieto et al. (2022).

The first explanation was consistent with Zazkis and Rouleau's (2018) study in which their PSTs also agreed that the order of computation will not affect the evaluation of expressions that were in the form of $a \times b \div c$. The design of Zazkis and Rouleau's (2018) is different from that of the present study because Zazkis and Rouleau (2018) focused on the operations of multiplication and division whereas the present study examined the operations of addition and subtraction in the calculation. In making connections between the procedures and the properties of operations, the PSTs realised that the left-to-right order was not a rigid rule to follow, and any order would result in the same answer due to the associative and inverse properties. To them, remembering the rule does not do much because they can make sense of the rule through making connections to the properties of operations. This finding supports the argument that stronger connections between properties of operations may lead to meaningful learning of the order of operations (Kalder, 2012; Lee et al., 2013; Watson, 2010; Zorin & Carver, 2015).

The second explanation for why both orders work was that both orders yield the same answer. In this regard, the link between the order convention and the properties of operations was established through completing procedures and comparing the final answer. Evitts (2004) used the notion of structural connection to describe this type of connection. Although a connection was identified, the underlying reasons for why both orders work were not uncovered in this explanation.

The third explanation that emerged in the data was dependent on the mnemonic BODMAS. In this explanation, the PSTs transformed the rules into another equivalent representation, that is an acronym. Researchers describe this connection as representational connection (Evitts, 2004; Rodríguez-Nieto et al., 2022). Like the second explanation, the reasons for the conventions were not discussed despite a representational connection being made.

The third explanation is again consistent with Zazkis and Rouleau's (2018) findings in which Canadian PSTs also used a mnemonic to support their choice of giving priority to division over multiplication. However, the PSTs in Zazkis and Rouleau's (2018) study reasoned based on the mnemonic BEDMAS whereas the PSTs in the present study used the mnemonic BODMAS. In countries that use PEMDAS or BOMDAS, such as the US, Glidden (2008) found that their teacher participants favoured multiplication before division. Even though different mnemonics were used, misinterpretations may still occur because PSTs are dependent on the ordered letters of the memorised mnemonic, presumably starting with subtraction and ending with operations within parentheses (Dupree, 2016). The findings of the present study extend the existing literature on the order of operations because limited research has been conducted in Malaysia. Drawing the sample from Malaysia is necessary because the professional knowledge of mathematics teachers may be slightly different from other countries due to differences in local institutions and cultural aspects. This type of explanation is probably indicative of a global phenomenon.

Another connection discussed in this study is relating the order conventions to the distributive property. Some may consider that the order of operations contradicts the distributive property and some may view them as complementary (Peterson, 2005). The PSTs of the present study explained the connections between the order of operations and the distributive property based on the characteristics of the procedures and the property. They viewed the distributive property as a way to rewrite the expression that subsequently allowed them to use the order of operations to simplify the expression. This finding is a contribution to the literature because there is limited research dealing with making connections between the order of operations and the distributive property. It is also worth examining the interconnectedness between the order of operations and other mathematical concepts such as the concept of set.

Making mathematical connections is a key goal in the learning of mathematics (NCTM, 2009) and researchers encouraged connection making in learning mathematics with understanding (Bossé, 2003; Cai & Ding, 2017). As discussed in Section 2.3.1 (p. 21), however, there is little research about how PSTs link the order of operations to the properties of operations (Zazkis, 2018). The findings about how PSTs made connections between the order of operations and the properties of operations represents a contribution to the existing literature and highlight the importance of making mathematical connections across the number domain. Future studies might explore how student learning is influenced by the PSTs' knowledge about mathematical connections.

6.2.3 The Order of Operations of Contextualised Problems

The ambiguity in the order of operations could be potentially avoided if students were given a contextualised problem rather than a numerical expression (e.g., Bay-Williams & Martinie, 2015; Cardone, 2015; Chang, 2019; Jeon, 2012). This section discusses the evidence of how the order of operations is recognised when PSTs mathematised contextualised problems. In particular, this section answers the third research question:

How do pre-service secondary mathematics teachers determine the order of operations of contextualised problems?

There is very little research about how the order of operations is determined if given a problem in context (Holm, 2021). In the present study, four contextualised problems were developed to examine this situation. The results suggest that in most cases when mathematising contextualised problems, the PSTs determined the order of operations based on the context of the problems. They were able to apprehend the underlying mathematical structure of the context and use the correct order of operations to reflect those underlying structures.

There was evidence that, to a certain extent, the ambiguity in the order of operations could be potentially avoided if participants were presented with contextualised problems instead of numerical expressions. For example, all the PSTs correctly used the order of operations $(a \div b \times c)$ to represent Problem 2a (as defined in Section 5.4, p. 133), but only 6 used the correct order when presented with the numerical expression of the same form, that is $a \div b \times c$ (as presented in Section 5.2.1, p. 93). The finding that the order of operations was determined based on the context of the problems (not depending on memorising the rules) and led to successful mathematisation is therefore a contribution to the existing literature. Future research may examine how the use of contextualised problems might help students make sense of order of operations.

On the other hand, the wording of the contextualised problems could potentially mislead PSTs. This is apparent when some of the PSTs misinterpreted the action of enlarging five times as multiplying by 5. The error they made was the application of the wrong operation (multiplication) and thus writing an incorrect expression to represent the contextualised Problem 2d. As contextualised problems may scaffold students' understanding (Clarke & Roche, 2018; Meyer et al., 2001), it is worth examining further how contextualised problems can be used in relation to the order of operations without confusing learners.

In some cases, the PSTs' explanations were based on BODMAS. Take contextualised Problem 2b for example, the PSTs simply referred to BODMAS as a reason for using parentheses at the start of the expression. This reflects the danger of using the acronym in relation to the order of operations as discussed in Section 5.2 (p. 93). In the long term, learning the order of operations without understanding may lead to misinterpretations or misuse (Dupree, 2016; Glidden, 2008; Pappanastos et al., 2002; Zazkis & Rouleau, 2018). PSTs not only need to mathematise problems correctly but also need to know why the order of the corresponding operations makes sense in the context of the problem. In doing so, teachers can help their students to make sense of convention, rather than rely on rote memorisation.

It is likely that the PSTs whose explanations were based on BODMAS actually understood the underlying mathematical structure of the context, even though this was not reflected in their explanations. For example, for Problem 2b, the PSTs probably realised that the difference between two numbers needed to be determined first thus they used subtraction at the start of the expression. As the subsequent operand was division, they added parentheses enclosing the subtraction to emphasise subtraction needed to be computed first. This suggests that the PSTs recognised the underlying mathematical structure of the context and used parentheses to change the standard procedure as they probably knew that the order of operations was doing division before subtraction. The finding about the use of parentheses is different from previous studies where the participants used parentheses as a strategy to facilitate calculations (e.g., Gunnarsson & Karlsson, 2014; Gunnarsson et al., 2016; Papadopoulos & Gunnarsson, 2018, 2020). The PSTs in the present study, however, used parentheses to change the standard procedure. Apparently, these PSTs were able to apprehend the context of the problem since they could use the correct order of operations to reflect the problem. However, they failed to explain their reasoning and simply referred to BODMAS.

Although contextualised problems could help with determining the orders or steps to solve the problems, a complication appeared when the PSTs wrote the steps in one single expression, without utilising parentheses. For example, Felix wrote a series of invalid mathematical equations to represent Problem 2c (see **Figure *5.48***, p. 141). He seemed to understand the context of the problem, but the only trouble was that he could not represent the problem in one single expression. I speculate that he might be combining steps quickly to get the answer and using the equal sign as a way of listing each step. Further research is needed to confirm the root cause of this complication.

In most cases, the PSTs were able to make sense of the hierarchy between operations and used a correct order of operations to reflect the contextualised problems. In some cases, however, there was evidence of unsuccessful mathematisations. These unsuccessful mathematisations of contextualised problems were related to an inability to apply relevant mathematical knowledge to formulate an expression for the context using correct symbolic form. A case in point is the contextualised Problem 2d that involved indices. It is acknowledged that this problem requires not only thinking of the order of operations but also the mathematics knowledge involved in the problem itself. As the present study does not examine if the PSTs consciously used the order of operations when transforming the index problem into mathematical form, future research may explore what role the order of operations plays in such problems, as well as the other index laws.

6.3 KNOWLEDGE OF STUDENTS' SENSE MAKING

This section discusses the results regarding the PSTs' interpretations of student written work involving the order of operations. This discussion contains answers to the following research question:

How do pre-service secondary mathematics teachers interpret students' written work involving the order of operations?

The results of this study demonstrate that PSTs interpreted students' written work mathematically, pedagogically, and through self-comparison (see **Table 5.6**, p. 150). Interpreting written work through mathematical approach means using knowledge of content to make conclusions or critique students' ways of solutions. Relying on pedagogical content knowledge, interpreting written work through pedagogical approach means drawing on knowledge about what students might commonly do and how a mathematical concept might generally be taught. In the present study, a new element of pedagogical approach that is not identified previously emerged from the data: making contextual interpretations to make sense of students' work. In this respect, context-based interpretations may include school practices, instructional time, parental and student expectations. Interpreting written work through self-comparison means searching for similarities and differences between PSTs' own solution and those of students.

Baldinger (2020) argues that PSTs at the beginning of the teacher training programme interpreted students' written work primarily through mathematical approach because the participants had limited opportunities to learn about students' common errors and students' mathematical thinking. Although the sample used in the present study was PSTs at the end of the teacher training programme, who apparently have relatively more pedagogical content knowledge, the present study also demonstrates a similar result as Baldinger (2020). This finding is surprising because with relatively more pedagogical content knowledge, the PSTs should be more likely to use different approaches to interpret students' written work. A possible explanation for this finding is that while PSTs might possess substantial pedagogical content knowledge to make sense of students' thinking, the knowledge was not exploited in interpreting students' work in this study.

There were a few instances of PSTs interpreted students' written work pedagogically. The PSTs drew on their knowledge about a common error of order of operations (i.e., perform multiplication before division) and a common teaching practice of order of operations (i.e., using an acronym) when interpreting the written work. It is important to note that the present study revealed one unanticipated finding about the pedagogical approach used to interpret students' written work. In addition to Baldinger's (2020) conceptualisation of the pedagogical approach, context-based interpretations also appeared to underpin this approach. For instance, both Eddy and Julie referred to the limited time of instruction when they were making sense of the students' error. The interpretation about instructional time is pertaining to the pedagogy aspect of teachers (Askew et al., 1997; Tiilikainen et al., 2019). Though this finding is based on interpretations made by only two PSTs and was not the most common interpretation appear in this study, the instances when it occurred provide insight into an additional element to describe interpretations that are pedagogical in nature. As the context-based interpretation may have the potential to contribute to the literature on PSTs' interpreting of students' written work, further research might examine the effect of this element on their classroom instruction.

Only on some occasions, the PSTs interpreted the students' written work through self-comparison. This is different from Baldinger's (2020) study because 75% of her participants engaged in making sense of students' work using the self-comparison approach. One potential explanation for this difference is that the written work in the present study involved order of operations and therefore relatively simple as compared to Baldinger's (2020) algebra and geometry student work. Given that the student work regarding the order of operations consisted of routine procedures, PSTs were able to promptly analyse the students' solutions without having to compute their own solutions. Unlike Baldinger's (2020) analysis of student work in algebra and geometry, it may be that the written work was more complex since there were more computations and properties involved. As a result, the PSTs in Baldinger's (2020) study were more likely to compute their own solutions and compare them with the students' solutions.

The results of the study indicate that the PSTs who used the self-comparison approach were not able to determine the students' errors. Although Baldinger (2020) claims that self-comparison is a useful approach in making inferential assertions about students' understanding, this approach is problematic to the PSTs in this study because they used their incorrect solutions to make sense of how the students used the order of operations. Consequently, the PSTs made the same error as the student. For instance, Howard misinterpreted the first written work based on his incorrect way of evaluation, that is performing multiplication before division. In this sense, it may be possible to say that the PST's knowledge about the order of operations is superficial at best. This finding is not unexpected, given that Leong et al. (2015) showed that Malaysian PSTs performed below the international average for both mathematics content knowledge and mathematics pedagogical content knowledge. Accordingly, the self-comparison approach is considered challenging as it requires PSTs' solutions to be correct in order to compare with students' work (Levin et al., 2009).

The existing literature in relation to analysing students' written work was largely restricted to describing students' procedures and checking students' answers for accuracy (e.g., Even & Tirosh, 1995; Gökkurt et al., 2013; Kiliç, 2011; Şahin et al., 2016; Shin, 2020; Tanisli & Kose, 2013; Tirosh, 2000), and emphasising on how interventions would support PSTs' interpretations towards students' work (e.g., Ergene & Bostan, 2022; Ivars et al., 2020; Monson et al., 2020; Sánchez-Matamoros et al., 2015). The present study differs from these past studies in that it analyses the approaches PSTs used in making interpretations about students' written work. This line of research is crucial because key features for supporting PSTs to develop abilities in interpreting students' sense making can be identified. It is acknowledged that the findings of the study could not be generalised to larger sets of research samples and to different mathematical task samples because the results are based on a small sample size of PSTs, who worked on the procedures of the order of operations and examined students' work with specific errors. However, the present study provides insight into the approaches that may help PSTs in successfully making sense of students' written work. At the same time, the study has highlighted difficulties that PSTs might experience when they lack the required content and pedagogical knowledge.

6.4 KNOWLEDGE OF TEACHING APPROACHES

This section discusses how PSTs planned to teach the order of operations, in line with the fifth research question:

How would pre-service secondary mathematics teachers plan to teach the order of operations?

There is very little research about the pedagogical aspect in relation to the order of operations (Ameis, 2011; Zazkis, 2018). In the present study, the results of interviews and lesson plans were conceived to accomplish this goal. Three main types of teaching approaches were identified as important in understanding the ways the PSTs planned to teach the order of operations. These approaches are transmission, connectionist, and discovery. The results show that some PSTs incorporated only a transmission approach and some combined more than one approach. No matter whether they used only one or more approaches, it is apparent that the PSTs were most likely to adopt the transmission approach.

Some PSTs used exclusively the transmission approach that emphasises a collection of routines or mathematical procedures in teaching. This result differs from that of Askew et al. (1997), whose study found that no teacher fitted exactly within this category. It may be that the transmission approach is more appropriate for the teaching of the order of operations because this topic is mainly about mathematical computations and procedures. For the PSTs who used this approach, they considered giving clear verbal explanations and using mnemonics as an effective means in memorising and applying the procedures. They believed that the order of operations was best learned through the paper and pencil method. However, disadvantages are more likely to arise if one single approach is used. For example, a lesson that is predisposed to the transmission approach might be inclined to give explanation and memorising procedures rather than encourage students to discover conceptual links between different mathematical concepts. Thus, utilising different teaching approaches is necessary dependent upon the nature of the concepts to be taught (Muir, 2008).

The results also indicate that some PSTs combined more than one teaching approach when planning the teaching of the order of operations. In addition to using the transmission approach as discussed in the previous paragraph, they also demonstrated a connectionist or discovery approach. In terms of the connectionist aspect, the PSTs were aware that the teaching of the order of operations could be linked with related mathematical ideas such as the properties of operations. However, they hardly made the links explicit in their planning. They predominantly planned for extensive dialogues to explore students' understandings and to challenge students' current levels of thinking. This result is different from Rowland et al.'s (2005) study in which the teacher trainees successfully made links with previous related lessons and between different strategies for subtraction. A potential reason for this difference is that PSTs are unlikely to make mathematical connections between multiplication and division (Ponte & Chapman, 2008) thus the relationships between the associative property of multiplication, the multiplicative inverse, and the left-to-right order were not explained by the PSTs in the present study.

With respect to the discovery aspect, the PSTs believed that all methods of calculation were equally acceptable if the correct answer was obtained. They encouraged students to work methods out themselves regardless of whether the methods were efficient. Furthermore, they strenuously argued for the importance of students' readiness prior to learning a new concept. In short, the PSTs who combined the transmission and discovery approaches tried to address both students' fluency and problem-solving. As consistent with the study of Askew et al. (1997), the PSTs who displayed the combination of transmission and discovery approaches planned for differentiated instruction in their teaching of the order of operations. They provided challenges for higher attaining students and prepared routines for lower attaining students. In other words, the PSTs recognised that the curriculum must be accessible to all students.

Only one PST did not display a transmission approach and used primarily connectionist and discovery approaches. In contrast to Swan and Swain's (2010) study, a significant number of teacher participants had moved away from the transmission approach. It is conjectured that most PSTs in the present study preferred the transmission approach because they could not recognise the limitations of this approach. In addition, they could be also influenced by their previous learning experience. This is clear where many PSTs responded that "I always follow the way my maths teacher taught me, she explained first then gave a worksheet", "My teacher gave lots of exercises to train me to be competent in solving expressions, I believe giving exercises for sure can train my students to be like me as well". The responses clearly demonstrate the PSTs preferred the transmission approach, which could be due to how they were taught about the procedures back in school. As effective learning is unlikely to occur if teaching is dependent only on lectures and exercises, teaching through stimulating students' reasoning and facilitating students to use their own strategies would be more beneficial in the long run (Alexander, 2006; Coben et al., 2007). It is thus proposed that pre-service development courses integrate different approaches into teaching routines, such as the order of operations to increase the involvement of PSTs.

6.5 SUMMARY OF RESEARCH IMPLICATIONS

This chapter discussed the results about PSTs' mathematical pedagogical content knowledge in relation to the order of operations. The results have six important implications for practice. The implications are discussed in turn.

First, the evidence that PSTs' exhibited flexibility in evaluating mathematical expressions based on the connections between the order of operations and relevant mathematical concepts alerts PST education to emphasise the sense making of the seemingly arbitrary convention. It is not new in the literature that a mathematical concept needs to be taught with understanding; this study provides insight that procedural rules also need to be made sense of. Making sense of the convention leads to an understanding that the left-to-right order can be justified using associative and inverse properties and this contributes to the flexibility in doing arithmetic operations. Having such flexibility is needed in arithmetic as it is fundamental to the development of algebra (Gray & Tall, 1994). As this study also highlighted misinterpretations of the order of operations, PST educators need to understand how PSTs make sense of the topic. This is important because we cannot expect PSTs to teach the order of operations correctly if the PSTs themselves erroneously interpreted the topic. Although the focus in this study was the order of operations, I suggest that other mathematical conventions, such as the use of superscript (-1) in mathematics (Kontorovich & Zazkis, 2017) and a^0 (Zazkis & Kontorovich, 2016), would also be worth further examination because making sense of mathematical conventions may lead learners to make connections that are rarely attended (Kontorovich & Zazkis, 2017). It is believed that being able to make sense of conventions may contribute to the flexibility in doing mathematics.

Second, the evidence that the PSTs obtained the correct answer but employed an incorrect order suggests that educators should not only focus on the correctness of the final answer, but also pay attention to the process and reasoning that led to the solution. Recognising and addressing situations where the correct outcome is achieved through an incorrect order allows educators to offer specific support and guidance aimed at cultivating a more precise and efficient approach. Consequently, this will facilitate the development of a deeper understanding of the mathematical content.

Third, the finding that PSTs made mathematical connections in relation to the order of operations adds to the empirical evidence for the use of different mathematical connections. Particularly, analogical connections as suggested by Rodríguez-Nieto et al. (2022) were evident when the PSTs linked a familiar situation (set or algebra) to the order of operations. The analogical connections made were either relevant or irrelevant. It is considered relevant when PSTs linked the order of operations to the concept of set and made sense of the left-to-right order correctly. It is irrelevant when PSTs connected the order of operations with quadratic equations and interpreted that the order of computations does not matter. The problem with making irrelevant mathematical connections is that mathematical ideas are built based on incorrect conceptions. Past research has documented mathematical connections in functions (García-García & Dolores-Flores, 2021; Hatisaru, 2022), geometry (Eli et al., 2013), and measurement (Gamboa et al., 2020), but none in order of operations, and therefore, the present study extends the work to understand how PSTs link the procedural rules within mathematics. Making connections for the procedural rules rather than memorising the rules is important because mathematics is “a connected and coherent enterprise” (Toh & Choy, 2021, p. 1) that needs to be made sense of. Moreover, effective teaching requires PSTs help students to make mathematical relationships that based on justification and proof (Simon, 2022), and therefore, PSTs must be able to recognise relevant mathematical connections before making them explicit and comprehensible to students. More importantly, they must be aware of the irrelevant connections that potentially mislead students when making sense of the order of operations.

Fourth, this study provides empirical evidence that the same interpretation of the order of operations can be achieved through using contextualised problems, as hypothesised in the literature. However, PSTs may misinterpret the contextualised problems, represent the problem with incorrect mathematical expressions, and use a memorised acronym after expressions were formed. For example, Audrey used a correct order of computations for this Problem 2d, but she did not express the order as a single valid mathematical expression. This implies that contextualised problems may lead to steps rather than orders in a concatenated expression. This finding informs our understanding about the sense-making of order of operations through contextualised problems and may be emphasised in mathematics instruction and learning. Future research might examine how the use of problems in context help students make sense of order of operations.

Fifth, the findings of the present study provide empirical support for Baldinger’s (2020) mathematical, pedagogical, and self-comparison approaches that are used to interpret students’ written work. In view that it is the biggest challenge for PSTs to understand what students’ think about a mathematical topic (Simon, 2022), knowing the ways PSTs used to analyse students’ work is crucial in order to identify key features to support PSTs in developing the ability to interpret students’ sense making. This study, therefore, serves as a reference for PST education to identify the most common approach that might appear among their students. At the same time PST educators can introduce these different approaches for their students to make sense of students’ written work more accurately. To continue discussing the fourth implication, I focus on the pedagogical approach to interpret students’ written work. As conceptualised by Baldinger (2020), the pedagogical approach draws on knowledge about what students might commonly do and how mathematics might generally be taught. In addition to this conceptualisation, an additional element yet unidentified in the literature was evident in the present study: context-based interpretation. In this respect, the context-based interpretation that appeared in this study was about instructional time. For example, both Eddy and Julie referred to the limited time of instruction when they were analysing the students' errors. The interpretation about instructional time is pertaining to the pedagogy aspect of teachers (Askew et al., 1997; Tiilikainen et al., 2019). It is worth examining further the potential of this new element of the pedagogical approach in influencing the analysis of students' written work. As the study also highlighted difficulties that PSTs might experience when they lack the required content and pedagogical knowledge to interpret students' written work, PST educators may incorporate more student work with different errors to support PSTs in making interpretations and therefore, develop PSTs' ability in interpreting students' sense making.

Finally, the findings of the present study extend the existing literature on the pedagogical aspect of the order of operations. As the literature lacks discussions on the pedagogical aspect of the order of operations (Ameis, 2011; Zazkis, 2018), this study filled this gap and found that the PSTs in this study were most likely to adopt the transmission approach when planning to teach the order of operations. Whilst Askew et al. (1997) showed in their study that transmission approach is a moderately effective approach to teach numeracy, the finding of the present study raises awareness about PSTs' ways for teaching, in particular when approaching procedural rules. As teaching the order of operations without understanding may lead to misunderstandings, linking the topic to properties of operations and/or refer to the hierarchical triangle may potentially reduce the risk of misinterpreting the order of operations. Therefore, it is proposed that pre-service mathematics education integrates different approaches into teaching procedural rules to increase the involvement of PSTs to appropriately connect relevant mathematical ideas together in sophisticated way (Tall, 2007).

Chapter 7: Conclusions

The present study investigated PSTs' mathematical pedagogical content knowledge of the order of operations. Specifically, the study examined mathematical pedagogical content knowledge in terms of knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches. By exploring these three knowledge components, it offers a holistic view of PSTs' expertise in teaching the order of operations.

This chapter is divided into several sections. Firstly, it presents a summary of the study, providing an overview of the research conducted. Secondly, it offers an overview of the contributions that the study makes, highlighting the novel insights generated through the study. Thirdly, the chapter addresses the limitations of the study, acknowledging the constraints in the research design and methodology. Additionally, the chapter provides suggestions for future research, identifying potential areas for future investigation. Lastly, the chapter concludes with closing remarks, summarising the significance of the study's contributions.

7.1 SUMMARY OF THE STUDY

The evidence demonstrating the ways in which PSTs make sense of mathematical procedures, specifically the order of operations in this study, is relatively limited. Furthermore, there is a scarcity of research focused on understanding the approaches PSTs employ to interpret students' sense making in relation to the order of operations. The literature also lacks discussions on PSTs' teaching approaches of the order of operations. These research gaps indicate the need for in-depth exploration of PSTs' mathematical pedagogical content knowledge. To address these research gaps, this study aimed to provide a comprehensive understanding of PSTs' knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches related to the order of operations. The investigation was guided by the following research questions:

1. *How do pre-service secondary mathematics teachers apply the order of operations to evaluate mathematical expressions?*
2. *How do pre-service secondary mathematics teachers interpret the connections between the order of operations and the properties of operations?*
3. *How do pre-service secondary mathematics teachers determine the order of operations of contextualised problems?*
4. *How do pre-service secondary mathematics teachers interpret students' written work involving the order of operations?*
5. *How would pre-service secondary mathematics teachers plan to teach the order of operations?*

The study employed a qualitative and interpretative methodological orientation to address these research questions. Using a case study with 11 participants, PSTs' mathematical pedagogical content knowledge was collected comprehensively through a questionnaire and interview tasks related to the order of operations. This method of data collection enabled a focus on the PSTs' justification for their computations and instructional choices. Additional data collected were the PSTs' lesson plans about the teaching of the order of operations. The data were theoretically and thematically analysed to describe the different knowledge components.

7.2 OVERVIEW OF THE CONTRIBUTIONS

This investigation contributes to mathematics education research in terms of the professional knowledge of pre-service mathematics teachers. In general, this study contributes to current research in the following ways: by providing a conceptual framework for presenting and analysing mathematical pedagogical content knowledge, by providing evidence of the knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches in relation to the order of operations, and by highlighting recommendations for PST education. This section discusses these contributions in turn.

This study makes a significant theoretical contribution to mathematics education research in terms of providing a conceptual framework to present and analyse mathematical pedagogical content knowledge of pre-service mathematics teachers. In particular, the conceptual framework developed in this study represents mathematical pedagogical content knowledge for teaching the order of operations and it would also apply to a range of other mathematical topics. The conceptual framework is illustrated in **Figure *7.1***.

Figure 7.1

The Conceptual Framework

This conceptual framework is an output derived specifically from this study, offering a novel and valuable contribution to the research literature. By progressively refining and enhancing the framework, I constructed this framework based on the unique research objectives, research questions, and context of this study. Drawing upon established theoretical frameworks and empirical findings, I carefully integrated three essential knowledge components. By doing so, this conceptual framework fills a significant gap in the existing literature by providing a comprehensive and systematic approach to understanding the nature and extent of mathematical pedagogical content knowledge utilised for teaching the order of operations. Its relevance lies in its ability to uncover both the strengths and limitations of PSTs' mathematical pedagogical content knowledge, specifically pertaining to the order of operations.

As highlighted in Chapter 3 (p. 47), the conceptual framework offered a useful structure and three relevant knowledge components in which to code and analyse mathematical pedagogical content knowledge of PSTs. Key elements underpinning the knowledge components were identified to enable a more compelling exploration of the professional knowledge of PST of mathematics. For example, in addition to examining fluency in doing mathematics, explanations underpinning PSTs' use of routine procedures were also included as an element of knowledge of content. Further to describing students' procedures and checking students' answers for accuracy, mechanisms by which PSTs interpret students' work were also incorporated as an element of knowledge of students' sense making. Apart from analysing how instructional activities are planned, the ways by which PSTs propose to eliminate students' common errors were also added as an element of knowledge of teaching approaches. The usefulness of this developed conceptual framework lies in its potential to explain how the key elements are related to each knowledge component and how the knowledge components are synthesised to produce mathematical pedagogical content knowledge of PSTs. Particularly, this conceptual framework extends beyond the immediate scope of this study, offering a unique perspective that initiates a broader discussion about various other mathematical concepts. By providing a comprehensive and systematic analysis, the conceptual framework adds value to the research literature by enhancing understanding, refining existing theories, and guiding future investigations in the field.

Contributing to the literature, this study has documented comprehensive insights into PSTs' knowledge that are relevant for the teaching of the order of operations. In terms of knowledge of content, the literature does not appear to critically discuss the operations of indices and parentheses in relation to the order of operations, and the present study allowed these procedures to be examined closely (see Section 5.2, p. 93). The study also demonstrates how PSTs apprehended the underlying mathematical structure of contextualised problems and used the correct order of operations to reflect those underlying structures (see Section 5.4, p. 133). Furthermore, the finding that some PSTs explained the order of operations based on the properties of operations represent a significant contribution to the existing literature about different mathematical connections across the number domain (see Section 5.3, p. 127). In terms of knowledge of students' sense making as discussed in Section 5.5 (p. 144), this study provides insight into the approaches that help PSTs make sense of students' written work. The difficulties that PSTs might experience when they lack the required content and pedagogical knowledge to interpret students' thinking were also highlighted. In terms of knowledge of teaching approaches, the literature lacks discussions on the pedagogical aspect of the order of operations from the perspective of PSTs. The present study addressed this gap and revealed that most PSTs preferred a transmission approach in planning the teaching of the topic (see Section 5.6, p. 150). These findings extend our understanding of the mathematical pedagogical content knowledge about the order of operations, and therefore can be emphasised in mathematics instruction and learning.

A key contribution to PST education includes raising awareness of PSTs' knowledge and understanding of the order of operations. In addition to emphasising PSTs' understanding of subject content, it is important for PST educators to be aware that teacher education courses should scaffold PSTs' abilities to interpret students' mathematical responses through a different lens (mathematical, pedagogical, and self-comparison as discussed in Section 5.5, p. 144) and to approach a mathematical concept in a variety of ways (transmission, connectionist, and discovery as discussed in Section 5.6, p. 150). PST educators need to recognise the current knowledge levels of PSTs and provide adequate discussions and opportunities for them to develop necessary knowledge and skills. They can replicate this study and determine for themselves whether their students have sufficient knowledge to teach the order of operations. If so, PST educators can use the framework developed in this study to analyse other mathematical concepts; if not, they must equip their students with the knowledge before the students enter the teaching profession. This contribution, therefore, allows PST educators to improve the quality of their students.

7.3 LIMITATIONS AND DIRECTIONS FOR FUTURE RESEARCH

Untangling the complexities of mathematical pedagogical content knowledge is worthy of continuing research (Lo, 2020). The present study, however, was limited to analysing PSTs' knowledge of the order of operations. It is acknowledged that the understanding of mathematical pedagogical content knowledge can only become clearer and richer when a wide range of mathematical concepts is examined in collegial ways. It would be illuminating to trial the conceptual framework of mathematical pedagogical content knowledge developed in this study with other mathematical concepts within the mathematics curriculum. Furthermore, future research might consider conducting longitudinal studies to explore the development of PSTs' mathematical pedagogical content knowledge over time. By tracking PSTs' understanding of mathematical concepts across different stages of their teacher education programs or professional development, researchers can gain insights into the progression and factors influencing the development of their professional knowledge.

The conduct of this study was restricted by Covid issues on the nature of data collection. First, data collection moved from in-person to a remote manner due to the whole country lockdown. This reduced the direct interactions between the participants and me in a research session. Second, the participants were a little rushed when completing the questionnaire and the clinical task-based interview via Zoom within 40-60 minutes. This might have impacted the quality of responses gathered. Third, network connectivity emerged as a barrier to remote data collection. The unstable internet connections during the interviews frustrated some of the participants. Fourth, classroom observations were not included due to school closures, and thus the investigation into real teaching practices were halted. Further research might employ in-person interviews and include classroom observations that potentially provide more information on the teacher knowledge needed to respond to students' ideas given in the classroom. In addition, the research settings could have been based on two separate sessions in which the first session involved administering the questionnaire and the second session involved conducting the clinical task-based interviews.

In order to avoid participants from feeling exhausted when completing the questionnaire and interviews in a single research setting, the current items were designed in a way that were less comprehensive. As this study was not primarily to analyse participants' ability or proficiency in using the order of operations, the items were sufficient to understand PSTs' mathematical pedagogical content knowledge of the order of operations. However, future research might consider using more comprehensive instruments containing all aspects of the order of operations. It may be rewarding to ask participants to provide numerical expressions they consider ambiguous and to discuss how to overcome the ambiguity. Instead of designing less comprehensive items, future research could adopt a mixed methods approach, combining quantitative and qualitative data collection methods. This allows for a more balanced and nuanced understanding of the research topic while providing flexibility in accommodating participant energy levels and avoiding exhaustion.

Some of the contextualised problems that PSTs were asked to represent with a single numerical expression were relatively complex. The case in point is the index problem (Problem 2d) as it requires the understanding of indices in addition to the understanding of the order of operations. The results of this problem suggest that the complexity of this problem might impinge on PSTs' responses. Further research might explore alternative contextualised problems designed to isolate PSTs' knowledge of parentheses and indices. Alternatively, future research could analyse the impact of problem complexity on the ability of PSTs to represent contextualised problems with single numerical expressions. This involves investigating how the level of complexity influences their understanding and ability to communicate mathematical ideas effectively.

The findings of the present study point to the need for further research into how the use of contextualised problems might help students make sense of order of operations. As presented in Section 6.2.3 (p. 171), the ambiguity in the order of operations could be potentially avoided when participants were presented with contextualised problems instead of numerical expressions. This finding supports suggestions from researchers to incorporate problems in context into the teaching of the order of operations (Bay-Williams & Martinie, 2015; Cardone, 2015; Chang, 2019; Jeon, 2012). In addition, researchers can compare the performance and strategies employed by PSTs when solving contextualised problems versus numerical expressions. Understanding the differences in performance can shed light on the effectiveness of using contextualised problems as a teaching tool for facilitating the understanding of the order of operations.

There was a noteworthy finding that emerged during PSTs interpreting students' written work but was not identified until the analysis. This finding was that PSTs made context-based interpretations when analysing students' work. Although this finding is based on responses of only two PSTs and is not the most common interpretation that appeared in the analysis, it is worth examining further. Further research might explore how this context-based interpretation influences PSTs' classroom instruction.

The utilisation of a Malaysian sample in this study is a limitation attributed to the variations in local institutions and cultural aspects. To address this limitation, future research could conduct comparative studies by selecting samples from diverse countries or regions. By doing so, it would be possible to explore the variations in professional knowledge among mathematics teachers and identify the specific factors that contribute to these differences. Additionally, such comparative studies would shed light on the impact of these variations on teaching practices and student outcomes.

Another limitation is the small sample size that was used for the data collection. Although the case study focused on exploring the unique and diverse ideas to achieve a deeper understanding about PSTs' mathematical pedagogical content knowledge, the sample size of 11 participants was slightly small. Future research might consider recruiting more participants and analysing each case or each PST in more depth so that the within case analysis can be performed more thoroughly.

This study was conducted with PSTs undertaking the same PST education courses at the same university. This represents a limitation of the present study because broad generalisation cannot be made. Future research might examine mathematical pedagogical content knowledge involving a more diverse sample of PSTs. By including a more diverse range of participants, such as different demographic backgrounds or educational settings, researchers can assess whether the observed trends and patterns hold true across a broader population. The next section presents a brief concluding remark.

7.4 CONCLUDING REMARKS

The work reported in this thesis is rooted in attention to what knowledge pre-service secondary mathematics teachers need in order to teach mathematics effectively. The knowledge includes knowledge of content, knowledge of students' sense making, and knowledge of teaching approaches that constitute mathematical pedagogical content knowledge. Within the present study, the complexities of mathematical pedagogical content knowledge were revealed from the perspective of 11 PSTs.

The contributions of this study are highly significant in the field of mathematics education research. The development of a conceptual framework for mathematical pedagogical content knowledge, specifically tailored to the teaching of the order of operations, offers a comprehensive and systematic approach to understanding the nature and extent of PSTs' knowledge in this area. The study provides valuable insights into the complexities of mathematical pedagogical content knowledge, shedding light on the areas where PSTs excel and areas that require further development. Moreover, the findings enhance our understanding of PSTs' knowledge of content, students' sense making, and teaching approaches, not only for the order of operations but also for other mathematical concepts. These contributions have practical implications for teacher education, enabling targeted interventions and professional development initiatives to improve the quality of PSTs and enhance mathematics instruction and learning.

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Appendices

Appendix A

Questionnaire

Section A
Biographical Information

Instruction: Complete your biographical information.

1. What is your highest academic qualification?

   ____

   ____

2. What is your first teaching area and second teaching area?

   First teaching area: ____________________

   Second teaching area: ____________________

3. What is your grade point average?

   ____

Section B
Pedagogical Content Knowledge of the Order of Operations

Instruction: Answer all the questions below.

1) Evaluate the following expressions without using a calculator. Write down precisely and clearly every step to reach your answer.

   a) $10 \div 5 \times 2 =$

   b) $4 \times 6 \div 3 =$

   c) $3 - 12 + 8 =$

   d) $4 + 17 - 7 =$

   e) $10 - 3^2 =$

   f) $2^{3^2} =$

   g) $8 - \big((-2) + 4\big) =$

   h) $(9 - 2)(7 - 4) =$

2) Write a mathematical expression to represent each of the following problems.

a) A tyre factory produces 6351 tyres every 3 days. How many tyres will the factory produce in 14 days?

b) There are 532 parking spots on the first level of a multi-level parking lot and the rest of the parking spots are distributed equally on the other 8 levels. How many parking spots are there on the top level if there are total of 1532 parking spots?

c) In a bookshop, paperback books cost $3 and hardback books cost $4 each. Alice buys six paperback and two hardback books. How much change will Alice receive from a $50 banknote?

d) Helen is enlarging a photo of 4cm width on her tablet screen. The width of the photo is doubled each time she enlarges the photo. What is the width of the photo on her screen if she enlarges the photo five times?

3) a) Lisa and Richard evaluate the expression $4 + 3 - 2$ differently. They ask you whose solution is correct.

Lisa's response	Richard's response
$4 + 3 - 2 = 7 - 2$ $= 5$	$4 + 3 - 2 = 4 + 1$ $= 5$

(i) Assess the students' responses.

(ii) Explain why Lisa and Richard can obtain the same answer even though they evaluate the expression in different ways.

b) Olivia and Desmond evaluate the expression $4(2+3)$ differently. They ask you whose solution is correct.

Olivia's response	Desmond's response
$4(2+3) = 4(5)$	$4(2+3) = 4(2) + 4(3)$
$= 20$	$= 8 + 12$
	$= 20$

(i) Assess the students' responses.

(ii) Explain why Olivia and Desmond can obtain the same answer even though they evaluate the expression in different ways.

Appendix B

Interview Tasks

Clinical Task-based Interview
Pedagogical Content Knowledge of the Order of Operations

Task 1

Michelle is a student in Grade 7. The following are her responses to two expressions.

a)

$$18 \div 3 \times 2 = \frac{18}{3 \times 2}$$
$$= \frac{18}{6}$$
$$= 3$$

b)

$$\frac{50 + 28}{5 + 7} = \frac{10 + 4}{1 + 1}$$
$$= \frac{14}{2}$$
$$= 7$$

i) Assess Michelle's responses.

ii) Why do you think she evaluated the expression that way? What misunderstandings/misconceptions is she likely to have?

iii) How would you teach to avoid students from making errors and misinterpreting the order of operations?

Task 2

Three teachers, Tracy, Connie, and Dickson each conducts an introductory lesson on the order of operations in three different 7th Grade classrooms. Their classroom practices are shown as below.

Teachers	Classroom practices
Tracy	Tracy writes the mnemonic, BIDMAS, on the board to introduce the order of operations. She explains that priority must be given to brackets before indices, followed by division and multiplication in order of appearance from left to right, then addition and subtraction in order of appearance from left to right. Students will need to memorise this mnemonic. She writes the expression $24 \div 6 \times 2$ on board and says “So, BIDMAS tells us that operations are to be carried out from left to right when we have division and multiplication in the expression.” She continues, “I have to divide 24 by 6 first (pointing at $24 \div 6$) to get 4 (write 4×2), then 4 times 2 so I get 8 (write 8). $24 \div 6 \times 2 = 4 \times 2 = 8$ Students are then given a number of expressions and told to evaluate based on BIDMAS. As Tracy moves around the class, she gives more expressions for students to evaluate.
Connie	In a lesson, Connie has set up 7 stations in which each station contains 3 cubes and 5 spinners. Connie requires students to calculate the total number of cubes, total number of spinners, total number of items in each station, and total number of items used for the whole lesson. Then she asks the students to form an expression that can be used to calculate the total number of items used for the whole lesson. Students work in pairs using a variety of methods. As they begin to complete the task, Connie brings the class together and invites students to provide the answers and explain the method used. The other students are attentive to these explanations. Students’ errors are discussed so that a more efficient method can be identified. Connie brings in the idea of brackets in helping students to refine their methods. They continue to discuss problems that involve different operations. Students develop an understanding of the order of operations through working on given problems and whole class discussion.
Dickson	Dickson organises students in groups. He gives a number of expressions to all the students and requires them to evaluate the expressions by any method. The expressions are as follows: $8 + 7 - 4 =$ $16 - 9 + 5 =$ $12 \times 6 \div 3 =$ $45 \div 5 \times 9 =$

	$4 + 3 \times 10 =$ $(25 - 3) \div 11 =$ $2 \times (24 \div 6) =$ $4 \times 3^2 =$ $(-2)^3 - 10 =$ Answers to all the expressions are provided. In groups, students evaluate the expressions but some of them obtain different answers. They are surprised that some expressions have several different answers. They compare their answers with the answers provided by Dickson then spend some time to recalculate and discuss within their groups. Based on evaluating the expressions, the students notice a pattern that leads them to generalise the correct order to perform computations.

a In your opinion, which classroom practice is the most effective practice to teach the topic? Why?

b Which classroom practice do you prefer to use? Why?

Appendix C

Interview Schedule

I. Opening

1. (Establish Rapport) My name is ___________. Thank you for volunteering to join this study.
2. (Purpose) First of all I would like to know about some of your background information. Then, you will need to complete a 3-item questionnaire. I am interested mainly in what is your knowledge about the order of operations as a per-service teacher. Then, I will ask you some questions based on your responses given in the questionnaire. After that, I will show you two tasks and ask you some questions based on the tasks.
3. (Motivation) I hope to use the information to inform the teaching of the topic.

(Transition: You may begin to fill in the questionnaire now.)

(Transition: Thank you for answering all the items. We will move to interview now.)

4. (Recording consent) What you say to me is very important. I want to make sure all my notes taken represent what you say, so I would like to take an audio recording and it is confidential and will not be shared around. Is this okay for you?

II. Body

Item 1a

1. In Item 1a, you responded ______. Please explain.

Item 1b

1. In Item 1b, your response is _____. Why?

Item 1c

1. In Item 1c, you wrote ______. Please elaborate. ....

Item 2a

1. In Item 2a, you wrote _______. Please elaborate on this.

Item 2b

1. In Item 2b, you used _____. What makes you responded that way? ....

Item 3a

1. For Item 3a, you stated that ____. Please explain further……...

(Transition: I am going to show you a student's written work on how she evaluated two mathematical expressions. You will have two minutes to look through the written work then I will ask you some questions.)

Task 1

Michelle is a student in Grade 7. The following are her responses to two expressions.

a)

$$18 \div 3 \times 2 = \frac{18}{3 \times 2}$$
$$= \frac{18}{6}$$
$$= 3$$

b)

$$\frac{50 + 28}{5 + 7} = \frac{10 + 4}{1 + 1}$$
$$= \frac{14}{2}$$
$$= 7$$

i) Assess Michelle's responses.

ii) Why do you think she evaluated the expression that way?
What misunderstandings is she likely to have?

iii) How would you teach to avoid students from making errors and misinterpreting the order of operations?

(Transition: I am going to show you another task about three different classroom practices. You will have five minutes to read through the task then I will ask you some questions.)

Task 2

Three teachers, Tracy, Connie, and Dickson each conducts an introductory lesson on the order of operations in three different 7^{th} Grade classrooms. Their classroom practices are shown as below.

Teachers	Classroom practices
Tracy	Tracy writes the mnemonic, BIDMAS, on the board to introduce the order of operations. She explains that priority must be given to brackets before indices, followed by division and multiplication in order of appearance from left to right, then addition and subtraction in order of appearance from left to right. Students will need to memorise this mnemonic. She writes the expression $24 \div 6 \times 2$ on board and says "So, BIDMAS tells us that operations are to be carried out from left to right when we have division and multiplication in the expression." She continues, "I have to divide 24 by 6 first (pointing at $24 \div 6$) to get 4 (write 4×2), then 4 times 2 so I get 8 (write 8). $24 \div 6 \times 2 = 4 \times 2 = 8$ Students are then given a number of expressions and told to evaluate based on BIDMAS. As Tracy moves around the class, she gives more expressions for students to evaluate.
Connie	In a lesson, Connie has set up 7 stations in which each station contains 3 cubes and 5 spinners. Connie requires students to calculate the total number of cubes, total number of spinners, total number of items in each station, and total number of items used for the whole lesson. Then she asks the students to form an expression that can be used to calculate the total number of items used for the whole lesson. Students work in pairs using a variety of methods. As they begin to complete the task, Connie brings the class together and invites students to provide the answers and explain the method used. The other students are attentive to these explanations. Students' errors are discussed so that a more efficient method can be identified. Connie brings in the idea of brackets in helping students to refine their methods. They continue to discuss problems that involve different operations. Students develop an understanding of the order of operations through working on given problems and whole class discussion.

Dickson	Dickson organises students in groups. He gives a number of expressions to all the students and requires them to evaluate the expressions by any method. The expressions are as follows: $8 + 7 - 4 =$ $16 - 9 + 5 =$ $12 \times 6 \div 3 =$ $45 \div 5 \times 9 =$ $4 + 3 \times 10 =$ $(25 - 3) \div 11 =$ $2 \times (24 \div 6) =$ $4 \times 3^2 =$ $(-2)^3 - 10 =$ Answers to all the expressions are provided. In groups, students evaluate the expressions but some of them obtain different answers. They are surprised that some expressions have several different answers. They compare their answers with the answers provided by Dickson then spend some time to recalculate and discuss within their groups. Based on evaluating the expressions, the students notice a pattern that leads them to generalise the correct order to perform computations. a In your opinion, which classroom practice is the most effective practice to teach the topic? Why? b Which classroom practice do you prefer to use? Why?

(Transition: Well, it has been a pleasure to interview you.)

III. Closure

1. (Maintain Rapport) I appreciate the time you took for this interview. Is there anything else you think would be helpful for me to know?
2. (Action to be taken) I should have all the information I need. Thank you.

Appendix D

EPU Approval

Unit Perancang Ekonomi
Economic Planning Unit
Jabatan Perdana Menteri
Prime Minister's Department
Setia Perdana 5 & 6, Kompleks Setia Perdana
Setia Perdana 5 & 6, Setia Perdana Complex
Pusat Pentadbiran Kerajaan Persekutuan
Federal Government Administrative Centre
62502 Putrajaya
MALAYSIA

Telefon : 603-8000 8000
Faks(fax) : 603-8888 3755
Laman Web : www.epu.gov.my

Our Ref.: EPU 40/200/19/3760(5)
Date: 4 January 2021

JIEW FUI FONG
School of Teacher Education and Leadership
Queensland University of Technology
Graduate Research Centre
GPO Box 2434
Brisbane QLD 4059 Australia
Email : jiewfuifong@gmail.com

APPLICATION TO CONDUCT RESEARCH IN MALAYSIA

With reference to your application, I am pleased to inform that your application to conduct research in Malaysia has been approved by the **Economic Planning Unit (EPU).** The details of the approval are as follows:

Researcher's name	:	**JIEW FUI FONG**
Passport No./ I.C No	:	**870222-52-5988**
Nationality	:	**MALAYSIAN**
Title of Research	:	**"PRE-SERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE: THE CASE OF THE ORDER OF OPERATIONS"**
Period of Research Approved	:	**1 year (14.01.2021 – 13.01.2022)**
Period of Researcher Pass	:	**1 year (14.01.2021 – 13.01.2022)** **Research Pass : 3380**

2. Please take note that the study should avoid sensitive issues pertaining to local values and norms as well as political elements. At all time, please adhere to the conditions stated by the code of conduct for researchers as attached.

3. The issuance of the research pass is also subject to your agreement on the following:

I. **Economics Planning Unit , Prime Minister Department**

a) To ensure submission of a brief summary of your research findings on completion of your research;

b) To submit three (3) copies of your final dissertation/publication; and

c) To return the research pass to the Researcher Pass Unit, Macroeconomics Division, Economics Planning Unit , Prime Minister Department.

4. Thank you for your interest in conducting research in Malaysia and wish you all the best in your future research endeavour.

Yours sincerely,

[redacted]

(MUHAMMAD JAWAD BIN TAJUDDIN)
Macroeconomics Division
for Director General
Economic Planning Unit
Prime Minister Department
Email : jawad.tajuddin@epu.gov.my/oridb@epu.gov.my
Tel : 03 88723254
Fax : 03 88883798

ATTENTION

This letter is only to inform you the status of your application and **cannot be used as a research pass.**

c.c. :
(for record)

Ketua Setiausaha
Kementerian Pengajian Tinggi
No. 2, Menara 2,
Jalan P5/6, Presint 5,
62200 Putrajaya
(u.p.: Setiausaha Bahagian Perancangan dan Penyelidikan Dasar)

Ketua Setiausaha
Kementerian Pendidikan Malaysia
Aras 1-4, Blok E8, Kompleks Kerajaan Parcel E
Pusat Pentadbiran Kerajaan Persekutuan
62604 Putrajaya
(u.p.: Bahagian Perancangan dan Penyelidikan Dasar Pendidikan)

Appendix E

Ethics Variation Approval

05/08/2021

Dear Prof Prof Lyndall English

We are pleased to advise that your request for variation has been reviewed and approved by the University Human Research Ethics Committee (UHREC) or delegated review body as meeting the requirements of the National Statement on Ethical Conduct in Human Research (2007, updated 2018).

Project title: Pre-service teachers' pedagogical content knowledge: The case of the order of operations

Approval number: 3893

Approved version: HE Migration Form 2021-3893-4820

Approval date: 05/08/2021

Approved amendments: Change to data collection. Addition of new data source (lesson plans).

Approved documents: Revised main form, revised participant information sheet, revised consent form.

Document Type	File Name	Date	Version
Historic Ethics/Biosafety Application	Approved Documents	03/08/2021	1
Historic Ethics/Biosafety Application	Main Form	03/08/2021	1
Historic Ethics/Biosafety Application	Participant Information Sheet	03/08/2021	1
Historic Ethics/Biosafety Application	Consent Form	03/08/2021	1

Additions do not alter the level of risk.

This approval is subject to the following standard conditions of approval as well as any additional conditions of approval indicated by the UHREC or delegated review body.

Additional conditions of approval:

- Nil

Kind regards,

Office of Research Ethics and Integrity
QUT Human Research Ethics Advisory Team | humanethics@qut.edu.au | +61 (0)7 3138 5123

Page 1 of 1

Appendix F

Approach Email

Subject Title:
Participate in a research study investigating pedagogical content knowledge

Dear Dr/Associate Professors/Professors

My name is Fui Fong Jiew from the School of Teacher Education and Leadership, Queensland University of Technology (QUT) and I'm doing a PhD into an understanding of pre-service teachers' pedagogical content knowledge.

I would like to invite third year and final year students enrolled in the Bachelor of Education (Mathematics) (Hons) and are studying mathematics as first or second teaching area to complete a questionnaire and semi-structured interview. These will be undertaken in a research session that will take around 40-60 minutes.

Please view the attached participant information sheet and consent form for further details on the study and how to participate.

If you are interested in participating, please contact me via email and returned a completed consent form. If you have any questions, please do not hesitate to contact me via email.

Please note that this study has been approved by the QUT Human Research Ethics Committee (approval number 2021000063).

Many thanks for your consideration of this request.

Fui Fong Jiew
PhD Student
fuifong.jiew@hdr.qut.edu.au

Prof Lyn English
Supervisor
31383329
l.english@qut.edu.au

Dr Timothy Lehmann
Associate Supervisor
31387341
t2.lehmann@qut.edu.au

School of Teacher Education and Leadership, Faculty of Education
Queensland University of Technology

Appendix G

Participant Information Sheet

PARTICIPANT INFORMATION FOR QUT RESEARCH PROJECT
– Overall Participation Information –

Pre-service teachers' pedagogical content knowledge: The case of the order of operations

QUT Ethics Approval Number 2021000063

Research team

Principal Researcher:	Fui Fong Jiew, PhD student
Associate Researcher(s):	Prof Lyn English, Principal Supervisor Dr Timothy Lehmann, Associate Supervisor
	School of Teacher Education and Leadership, Faculty of Education

Why is the study being conducted?

This research project is being undertaken as part of a PhD study for Fui Fong Jiew.

Pedagogical content knowledge is a form of teacher knowledge. This knowledge is important in teaching and pre-service teacher education. The purpose of this project is to explore pre-service teachers' pedagogical content knowledge in relation to the order of operations in mathematics education.

You are invited to participate in this research project because you are a third year or final year student enrolled in the Bachelor of Education (Mathematics) (Hons) and you are studying mathematics as your first or second teaching area.

What does participation involve?

Your participation will involve three components: (i) a questionnaire; (ii) an audio recorded interview; and (iii) lesson plans. The questionnaire and the interview will be undertaken in one research session that will take around 40-60 minutes of your time. Your previously developed lesson plans related to the teaching of the order of operations will be collected after the interview.

The questionnaire will ask things like:

- For each of the following student work samples, determine whether there were errors. If there were errors, provide a correct working.

The interview will ask things like:

- Can you think of other ways to teach the order of operations to generate effective student learning?
- Based on the dialogue between a teacher and two students discussing ways of evaluating an expression, is it true that no matter which operation we do first, we will always get the same answer? Please explain.

Your participation in this research project is entirely voluntary. If you agree to participate you do not have to complete any question(s) you are uncomfortable answering. Your decision to participate or not participate will in no way impact upon your current or future relationship with QUT (for example your grades). If you do agree to participate you can withdraw from the research project before, during, or up to two weeks after the data collection research session without comment or penalty by emailing Fui Fong Jiew (details below). Any information already obtained that can be linked to you will then be destroyed. You will be able to review your responses before submitting the questionnaire and review a transcript of your responses after the interview. Reviewing the transcript might take another 30-40 minutes of your time.

What are the possible benefits for me if I take part?

It is expected that this research project may not benefit you directly beyond you feeling satisfaction in having contributed to the development of new knowledge. Your responses will help us to better understand pedagogical content knowledge in mathematics education. A brief summary of the outcomes of the study will be sent to you via email when the study is completed.

To recognise your contribution should you choose to participate, the research team is offering a AUD20 Starbucks gift card as a token of appreciation for your time. The gift card will be given to you at the end of the research session.

What are the possible risks for me if I take part?

There are minimal risks associated with your participation in this research project. These include some likelihood for you to experience inconvenience and mild discomfort. To minimise the risk of inconvenience, the research session will be scheduled at a mutually convenient time and place and your schedule will be prioritised. Sufficient time will be given to send your previously developed lesson plans to Fui Fong Jiew (details below). To minimize the risk of mild discomfort, the project's purpose and some potential questions are outlined in the participant information sheet. Time will be given for you to independently decide your participation. You will be able to stop, pause, or reschedule your research session if required.

QUT provides for limited free psychology, family therapy or counselling services for research participants of QUT research projects who may experience discomfort or distress as a result of their participation in the research. Should you wish to access this service please call the Clinic Receptionist on **07 3138 0999** (Monday–Friday only 9am–5pm), QUT Psychology and Counselling Clinic, 44 Musk Avenue, Kelvin Grove, and indicate that you are a research participant. Alternatively, Lifeline provides access to online, phone or face-to-face support, call **13 11 14** for 24 hour telephone crisis support. If you are aged up to 25, you can also call the Kids Helpline on **1800 551 800**.

What about privacy and confidentiality?

All responses are coded, i.e., it will be possible to re-identify you. A re-identifying code stored separately to personal information, the code will only be accessible to the research team, and the code plus identifying information will be destroyed based on the University Retention and Disposal Schedule. Any data collected as part of this research project will be stored securely as per QUT's Management of research data policy. Consent forms will be stored for 15 years and then destroyed. Data will be stored for a minimum of 5 years and can be disclosed if it is to protect you or others from harm, if specifically required by law, or if a regulatory or monitoring body such as the ethics committee requests it.

Any personal information that could potentially identify you will be removed or changed before files are shared with other researchers or results are made public. You will not be named throughout the study, in the transcripts, and in the publication of results.

As the research project involves an audio recording:

- You will have the opportunity to verify your responses prior to final inclusion – that is, we will send your interview transcript back to you for your review. You will have a two-week window in which to return the transcript via email.
- The recording will not be used for any other purpose.
- Only the named researchers will have access to the recording.
- It is not possible to participate in the research project without being recorded.

How do I give my consent to participate?

We would like to ask you to sign a consent form (enclosed) to confirm your agreement to participate.

Valid consent can be returned via email in one of the four possible ways: (i) by printing, signing, scanning, and returning the consent form as an email attachment; (ii) by inserting an electronic signature and returning the consent form as an email attachment; (iii) by sending a reply email indicating your consent to participate in the text of the email.; (iv) by printing, signing and photographing the consent form and sending this as an attachment to Fui Fong Jiew (details below).

What if I have questions about the research project?

If you have any questions or require further information please contact one of the listed researchers:

Fui Fong Jiew	fuifong.jiew@hdr.qut.edu.au	
Prof Lyn English	l.english@qut.edu.au	31383329
Dr Timothy Lehmann	t2.lehmann@qut.edu.au	31387341

What if I have a concern or complaint regarding the conduct of the research project?

QUT is committed to research integrity and the ethical conduct of research projects. If you wish to discuss the study with someone not directly involved, particularly in relation to matters concerning policies, information or complaints about the conduct of the study or your rights as a participant, you may contact the QUT Research Ethics Advisory Team on +61 7 3138 5123 or email humanethics@qut.edu.au.

Thank you for helping with this research project. Please keep this sheet for your information.

Appendix H

Consent Form

QUT	**CONSENT FORM FOR QUT RESEARCH PROJECT** **– Interview –**

Pre-service teachers' pedagogical content knowledge: The case of the order of operations

QUT Ethics Approval Number 2021000063

Research team

Fui Fong Jiew	fuifong.jiew@hdr.qut.edu.au	
Prof Lyn English	l.english@qut.edu.au	31383329
Dr Timothy Lehmann	t2.lehmann@qut.edu.au	31387341

Statement of consent

By signing below, you are indicating that you:

- Have read and understood the information document regarding this research project.
- Have had any questions answered to your satisfaction.
- Understand that if you have any additional questions you can contact the research team.
- Understand that you are free to withdraw without comment or penalty.
- Understand that if you have concerns about the ethical conduct of the research project you can contact the Research Ethics Advisory Team on +61 7 3138 5123 or email humanethics@qut.edu.au.
- Understand that the research project will include three components: (i) a questionnaire; (ii) an audio recorded interview; and (iii) lesson plans
- Understand that it is not possible to participate in the interview component without being audio recorded.
- Agree to participate in the research project.

Name ______________________________

Signature ______________________________

Date ______________________________

Preferred email address for study correspondence ______________________________

Please return the signed consent form to the researcher.