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A fast time stepping Legendre spectral method for solving fractional Cable equation with smooth and non-smooth solutions

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Abstract:
To improve the calculation accuracy and efficiency, in this article, we develops a fast time stepping Legendre spectral method for solving fractional Cable equation, where in temporal direction the time stepping method is utilized and the spatial variable is discretized by Legendre spectral method. The time stepping method is used to approximate fractional order derivative, and its convergence accuracy in time is $O(\tau^2)$. The fast algorithm is applied to the time stepping method and it can reduce the computational complexity from $O(M^2)$ to $O(M \log M)$, where M denotes the number of time stepping. For non-smooth solutions, we deal with the initial singularity by adding correction terms. We also analyze the numerical stability and convergence in detail. Numerical experiments confirm our theoretical analysis and efficiency of the fast algorithm.

Keywords: Time stepping method, Legendre spectral method, Stability and convergence, Fast algorithm, Smooth and non-smooth solutions

1. Introduction

Fractional differential equations (FDEs) have aroused the interest of many researchers for their ability to describe accurately long-range cumulative efforts [1, 2, 3]. Actually, FDEs are widely employed in many fields, such as anomalous diffusion, image process, biological system, seismic singularity analysis and non-Newtonian fluid mechanics [4, 5, 6, 7]. The model of FDEs shows better description of physical phenomenons compared with the traditional model.

As well known, the solutions of FDEs are not expressed in a closed form in most cases, so developing efficient numerical methods to solve FDEs is crucial. Analysis on solving FDEs by numerical method has been a trendy, such as finite difference method [8, 9, 10, 11], finite element method [12, 13, 14, 15], finite volume method [16, 17, 18], spectral method [19, 20, 21, 22] and other methods [23, 24].

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In this paper, we consider the following fractional Cable equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_{0,t}^{1-\gamma_1} \dfrac{\partial^2 u}{\partial x^2} - D_{0,t}^{1-\gamma_2} u + f(x,t), \\ u(x,0) = \Phi_0(x),\ u_t(x,0) = \Phi_1(x),\ 0 \le x \le L, \\ u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \end{cases} \tag{1.1}$$

where $0 < x < L,\ t > 0,\ 0 < \gamma_1, \gamma_2 < 1$. $D_{0,t}^{1-\gamma_i},\ i = 1, 2$ are the Caputo fractional derivatives with respect to t, which can be defined by

$$D_{0,t}^{\gamma} u(x,t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-1-\gamma} \frac{\partial^n u(x,s)}{\partial s^n} \mathrm{d}s,\ 0 \le n-1 < \gamma < n,\ n \in N.$$

Fractional Cable equation is essential in neuronal dynamics, which models the anomalous electrical diffusion of neurons [25, 26]. Many scholars investigated Fractional Cable equation [27, 28]. Sweilan et al. proposed a novel numerical scheme for solving 2D fractional Cable equation, whose convergence rate is $O(\tau + h_x^4 + \xi h_y^4)$ [29]. Recently, Liu et al. utilized $L1$ scheme in time and implicit compact difference scheme in space, and presented a fast technique to reduce computational complexity from $O(N^2)$ to $O(N \log^2 N)$ [30], where N denotes $1/\tau$, its convergence accuracy in temporal direction is $O(\tau^{\min\{1+\gamma_1, 1+\gamma_2\}})$. Lin et al. conducted spectral method in spatial discretization and its convergence accuracy is $O(\tau^{2-\max\{\alpha,\beta\}} + \tau^{-1} N^{-m})$[31], where N denotes the degree of polynomial, m is the regularity of its solution. Mohebbi et al. used a scheme with second order accuracy in time and RBF meshless method in space [32]. Compared their analysis, we apply a fast time stepping Legendre spectral method for solving fractional Cable equation and our highlights focus on three aspects:

- We utilize a time stepping method in temporal direction and spectral method in spatial direction, and our convergence rate is $O(\tau^2 + N^{-m})$, where N denotes the degree of polynomial, τ is the time step, m is the regularity of solution.
- We apply a fast algorithm to overcome the dense computation caused by fractional derivatives, which can help us reduce the computational work from $O(M^2)$ to $O(M \log M)$.
- We add correction terms to improve the convergence rate for non-smooth solutions.

We outline this paper as follows. In Section 2, we make some preparations for following analysis. We apply a time stepping Legendre spectral method to formulate a fully discrete scheme in Section 3. In Section 4, the stability and convergence of the fully discrete scheme are rigorously proved. In Section 5, a scheme with corrections terms is given for solving fractional Cable equation with non-smooth solutions. In Section 6, a fast algorithm is implemented to accelerate the computation. In Section 7, numerical experiments confirm our theoretical analysis and efficiency of the fast algorithm. Finally, we make a conclusion.

2. Preparations

In this section, we give some definitions and lemmas for the following analysis. Define functional spaces by

$$H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\},$$
$$H^m(\Omega) = \{v \in L^2(\Omega), \partial_x^i v \in L^2(\Omega),\ \forall i \le m,\ i \ge 0\},$$

where $\Omega \in (0,1)$, $L^2(\Omega)$ is Lebesgue space, which is the space consisted of quadratically Lebesgue integrable functions, and its inner product and norm are defined following

$$(u, v) = \int_\Omega uv\mathrm{d}x,\ ||v|| = \sqrt{(v, v)}$$

Let $P_N(\Omega)$ be a space, which is composed of polynomials of degree less than $N-1$, and $P_N^0 \triangleq \{w \in P_N(\Omega)|w(\partial\Omega) = 0\}$.

Denote $\pi_N^{1,0}$ be the H_0^1 - orthogonal projection operator from $H_0^1(\Omega)$ into P_N^0, such that for all $w \in H_0^1(\Omega)$

$$(\partial_x \pi_N^{1,0} w, \partial_x v) = (\partial_x w, \partial_x v),\ v \in P_N^0.$$

We assume the solution of (1.1) has the following expression [12, 33]

$$u(x,t) = \Phi_0 + \Phi_1 t + c_2 t^{\sigma_2} + c_3 t^{\sigma_3} + \cdots = \Phi_0 + \Phi_1 t + \sum_{j=2}^{m} c_j t^{\sigma_j} + \phi(x,t), \quad (2.1)$$

define $\sigma_1 = 1$ and $\sigma_i < \sigma_{i+1},\ i \le m-1$. $\phi(x,t)$ is smooth enough for x and t, and $c_j \in H_0^1(\Omega) \cap H^m(\Omega)$, $\exists\ c_j \neq 0$ for $j = 2, 3, \cdots, m$.

σ ($\sigma \ge 1$) is used to indicate the regularity of the solution, defined by:

$$\sigma = \begin{cases} \sigma_2, & \Phi_1 = 0 \\ 1, & \text{otherwise} \end{cases} \quad (2.2)$$

Lemma 2.1. [7] *For the projection operator $\pi_N^{1,0}$, let $u \in H_0^1(\Omega) \cap H^m(\Omega)$, it is known that the following projection estimation holds*

$$||u - \pi_N^{1,0} u|| \le CN^{-m}||u||. \quad (2.3)$$

Lemma 2.2. *(Grönwall's inequality)* [20] *Let k, B and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integer $\mu \ge 0$, be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \le k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B,\ n \ge 0.$$

Suppose that $k\gamma_\mu < 1$, for all μ, set $\sigma_\mu = (1 - k\gamma_\mu)^{-1}$, then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \le \exp(k\sum_{\mu=0}^{n} \sigma_\mu\gamma_\mu)(k\sum_{\mu=0}^{n} c_\mu + B),\ n \ge 0.$$

3. Fully discrete scheme

Divide $[0,T]$ into M segments evenly, $\tau=\frac{T}{M}$, $t_k=k\tau,\ k=0,1,2,\cdots,M$, $u^k\triangleq u(t_k)=u(k\tau)$. The time stepping method is the discretization for fractional derivative and first derivative utilizing the novel formulas with shifted parameter θ, see Lemmas 3.1 and 3.2.

Lemma 3.1. [12] *Caputo fractional derivative $D_{0,t}^{\gamma}u$ $(0<\gamma<1)$ is approximated at $t_{n-\theta}$*

$$\begin{aligned} D_{0,t}^{\gamma}u(t_{n-\theta}) &= D_{\tau,\gamma}^{n,\theta}u+E_{n-\theta}^{(\gamma)} \\ &= \tau^{-\gamma}\sum_{k=0}^{n}\omega_{n-k}^{(\gamma)}(u^k-u^0)+E_{n-\theta}^{(\gamma)}, \end{aligned} \tag{3.1}$$

where $u^0=\Phi_0$, $E_{n-\theta}^{(\gamma)}=O(t_{n-\theta}^{\sigma-\gamma-2}\tau^2)$, coefficients $\omega_k^{(\gamma)}$ are as follows:

$$\omega_k^{(\gamma)}=\begin{cases} 2/[2(1+\theta)-\gamma], & k=0 \\ 4W_1^1/[2(1+\theta)-\gamma]^2, & k=1 \\ (W_k^1\omega_{k-1}^{(\gamma)}+W_k^2\omega_{k-2}^{(\gamma)})/k/(1-\gamma/2+\theta), & k\geq 2 \end{cases} \tag{3.2}$$

$$\begin{aligned} W_k^1 &= -\gamma+(\gamma-1)(\gamma/2-\theta)-(k-1)(\gamma-2\theta-1), \\ W_k^2 &= -(\gamma-1)(\gamma/2-\theta)-(k-2)(\theta-\gamma/2). \end{aligned}$$

Remark 1. The weights $\omega_k^{(\gamma)}$ are the expansion coefficients of the generating function $\omega(\xi,\gamma)$ which is defined by

$$\omega(\xi,\gamma)=\sum_{k=0}^{\infty}\omega_k^{(\gamma)}\xi^k=\frac{(1-\xi)^{\gamma}}{1-(\frac{\gamma}{2}-\theta)(1-\xi)},\ \gamma\in(0,1),\ \theta\in(\frac{\gamma-1}{2},1]$$

Lemma 3.2. [12, 34] *First order derivative u_t is approximated at $t_{n-\theta}$*

$$\begin{aligned} & u_t(t_{n-\theta})=u_{\tau,\theta}^{n}+E_{n-\theta}^{(1)} \\ &= \begin{cases} \dfrac{u^1-u^0}{\tau}+E_{1-\theta}^{(1)}, & n=1 \\ \dfrac{3-2\theta}{2\tau}u^n-\dfrac{2-2\theta}{\tau}u^{n-1}+\dfrac{1-2\theta}{2\tau}u^{n-2}+E_{n-\theta}^{(1)}, & n\geq 2 \end{cases} \end{aligned} \tag{3.3}$$

where $E_{n-\theta}^{(1)}=O(t_{n-\theta}^{\tilde{\sigma}-3}\tau^2)$, $\tilde{\sigma}=min\{\sigma_2,\sigma_3,\cdots\}\setminus\{\ 2\ \}$.

Using (3.1) and (3.3), we discretize (1.1) in the temporal direction
Case $n=1$:

$$\begin{aligned} \frac{u^1-u^0}{\tau} =&\tau^{\gamma_1-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_1)}\Delta(u^k-u^0) \\ &-\tau^{\gamma_2-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_2)}(u^k-u^0)+f^{1-\theta}+E_{1-\theta}. \end{aligned} \tag{3.4}$$

Case $n \geq 2$:

$$\begin{aligned}
&\frac{3-2\theta}{2\tau}u^n - \frac{2-2\theta}{\tau}u^{n-1} + \frac{1-2\theta}{2\tau}u^{n-2} \\
=&\tau^{\gamma_1-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_1)}\Delta(u^k-u^0) \\
&-\tau^{\gamma_2-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_2)}(u^k-u^0)+f^{n-\theta}+E_{n-\theta}.
\end{aligned} \tag{3.5}$$

where $f^{n-\theta}=f(x,t_{n-\theta})$ and the error term $E_{n-\theta}$ is

$$E_{n-\theta}=O(t_{n-\theta}^{\sigma+\gamma_1-3}\tau^2)+O(t_{n-\theta}^{\sigma+\gamma_2-3}\tau^2)+O(t_{n-\theta}^{\tilde{\sigma}-3}\tau^2). \tag{3.6}$$

Then we can get the fully discrete scheme by applying Legendre spectral method in space. The Legendre spectral method in weak formulation is to find $u_N^k \in P_N^0$ (for $\forall v \in P_N^0$), such that

Case $n = 1$:

$$\begin{aligned}
\left(\frac{u_N^1-u_N^0}{\tau},v\right) =& -\tau^{\gamma_1-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_1)}\left(\partial_x u_N^k-\partial_x u_N^0,\partial_x v\right) \\
&-\tau^{\gamma_2-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_2)}\left(u_N^k-u_N^0,v\right)+\left(f^{1-\theta},v\right),
\end{aligned} \tag{3.7}$$

Case $n \geq 2$:

$$\begin{aligned}
&\frac{3-2\theta}{2\tau}\left(u_N^n,v\right)-\frac{2-2\theta}{\tau}\left(u_N^{n-1},v\right)+\frac{1-2\theta}{2\tau}\left(u_N^{n-2},v\right) \\
=&-\tau^{\gamma_1-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_1)}\left(\partial_x u_N^k-\partial_x u_N^0,\partial_x v\right) \\
&-\tau^{\gamma_2-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_2)}\left(u_N^k-u_N^0,v\right)+\left(f^{n-\theta},v\right),
\end{aligned} \tag{3.8}$$

with $u_N^0=\pi_N^{1,0}u^0$, where $n=1,2,\cdots,M$.

4. Stability and convergence analysis

Lemma 4.1. [12] $\omega_k^{(\gamma)}$ *are defined in (3.1). For any vector* $(u^1,..,u^M)\in\mathbb{R}^M$ *with* $M\geq 1$*, it satisfies that*

$$\sum_{k=1}^{M}\sum_{i=1}^{k}\omega_{k-i}^{(\gamma)}u^iu^k \geq 0, \tag{4.1}$$

where $\gamma\in(0,1)$*,* $\theta\in(\frac{\gamma-1}{2},1]$.

Lemma 4.2. [12] $u_{\tau,\theta}^j$ *are defined in (3.3). For any vector* $(u^1, ..., u^M) \in \mathbb{R}^M$ *with* $M \geq 2$*, it satisfies that*

$$\sum_{j=1}^{M} u^j u_{\tau,\theta}^j \geq \frac{1}{4\tau}(u^M)^2 - \frac{1}{2\tau}(u^1)^2, \tag{4.2}$$

where $\theta \in [0, 1]$.

Theorem 4.3. *The schemes (3.7) and (3.8) derived by our method are unconditionally stable, and they have the following estimate*

$$||u_N^M|| \leq C(||u_N^0|| + \max_{0\leq j\leq M} ||f^j||). \tag{4.3}$$

Proof. We approximate u^0 with u_N^0, which holds $||u_N^0|| \leq ||u^0||$. For simplicity, define $v_N^n \triangleq u_N^n - u_N^0$ and we can derive by (3.1), (3.3)

$$\begin{aligned}
&\left(\frac{v_N^1 - v_N^0}{\tau}, v\right) = \left(\frac{u_N^1 - u_N^0}{\tau}, v\right), \\
&\frac{3-2\theta}{2\tau}\left(v_N^n, v\right) - \frac{2-2\theta}{\tau}\left(v_N^{n-1}, v\right) + \frac{1-2\theta}{2\tau}\left(v_N^{n-2}, v\right) \\
=&\frac{3-2\theta}{2\tau}\left(u_N^n, v\right) - \frac{2-2\theta}{\tau}\left(u_N^{n-1}, v\right) + \frac{1-2\theta}{2\tau}\left(u_N^{n-2}, v\right), \\
&D_{\tau,\gamma}^{n,\theta} v_N^n = D_{\tau,\gamma}^{n,\theta} u_N^n.
\end{aligned} \tag{4.4}$$

Replacing v with v_N^n in (3.8) and utilizing (4.4), we obtain

$$\begin{aligned}
&\frac{3-2\theta}{2\tau}\left(v_N^n, v_N^n\right) - \frac{2-2\theta}{\tau}\left(v_N^{n-1}, v_N^n\right) + \frac{1-2\theta}{2\tau}\left(v_N^{n-2}, v_N^n\right) \\
=&-\tau^{\gamma_1-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_1)}\left(\partial_x v_N^k, \partial_x v_N^n\right) \\
&-\tau^{\gamma_2-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_2)}\left(v_N^k, v_N^n\right) + \left(f^{n-\theta}, v_N^n\right),
\end{aligned} \tag{4.5}$$

then, we replace n with j and sum both sides for j from 1 to M ($M \geq 2$), we get

$$\begin{aligned}
&\left(\frac{v_N^1}{\tau}, v_N^1\right) + \frac{3-2\theta}{2\tau}\sum_{j=2}^{M}\left(v_N^j, v_N^j\right) \\
&-\frac{2-2\theta}{\tau}\sum_{j=2}^{M}\left(v_N^{j-1}, v_N^j\right) + \frac{1-2\theta}{2\tau}\sum_{j=2}^{M}\left(v_N^{j-2}, v_N^j\right) \\
=&-\tau^{\gamma_1-1}\sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_1)}\left(\partial_x v_N^k, \partial_x v_N^j\right) \\
&-\tau^{\gamma_2-1}\sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}\left(v_N^k, v_N^j\right) + \sum_{j=1}^{M}\left(f^{j-\theta}, v_N^j\right).
\end{aligned} \tag{4.6}$$

Using the Lemmas 4.1 and 4.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left(\frac{v_N^1}{\tau}, v_N^1\right) + \frac{3-2\theta}{2\tau}\sum_{j=2}^{M}\left(v_N^j, v_N^j\right) - \frac{2-2\theta}{\tau}\sum_{j=2}^{M}\left(v_N^{j-1}, v_N^j\right)\\ &+ \frac{1-2\theta}{2\tau}\sum_{j=2}^{M}\left(v_N^{j-2}, v_N^j\right) \geq \frac{1}{4\tau}||v_N^M||^2 - \frac{1}{2\tau}||v_N^1||^2,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\tau^{\gamma_1-1}\sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_1)}\left(\partial_x v_N^k, \partial_x v_N^j\right) = \tau^{\gamma_1-1}\int_\Omega \sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_1)}\partial_x v_N^k \partial_x v_N^j \mathrm{d}x \geq 0,\\ &\tau^{\gamma_2-1}\sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}\left(v_N^k, v_N^j\right) = \tau^{\gamma_2-1}\int_\Omega \sum_{j=1}^{M}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)} v_N^k v_N^j \mathrm{d}x \geq 0,\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\sum_{j=1}^{M}\left(f^{j-\theta}, v_N^j\right) \leq \frac{1}{2}\sum_{j=1}^{M}(||f^{j-\theta}||^2 + ||v_N^j||^2)\\ \leq &\frac{1}{2}\sum_{j=1}^{M}||v_N^j||^2 + C\sum_{j=1}^{M}(||f^j||^2 + ||f^{j-1}||^2)\\ \leq &\frac{1}{2}\sum_{j=1}^{M}||v_N^j||^2 + C\sum_{j=0}^{M}||f^j||^2.\end{aligned} \tag{4.9}$$

Ignore the nonnegative terms, we get

$$||v_N^M||^2 \leq 2||v_N^1||^2 + 2\tau\sum_{j=1}^{M}||v_N^j||^2 + C\tau\sum_{j=0}^{M}||f^j||^2. \tag{4.10}$$

Similarly, for $n = 1$, with (3.7) and Cauchy-Schwarz inequality we have

$$\left(\frac{1}{\tau} - \frac{1}{2}\right)||v_N^1||^2 \leq C(||f^0||^2 + ||f^1||^2), \ \tau < 2 \tag{4.11}$$

Using Grönwall's inequality, we obtain

$$||v_N^M||^2 \leq C(\tau\sum_{j=0}^{M}||f^j||^2). \tag{4.12}$$

where C is independent of n and τ.

Finally, using the triangular inequality $||u_N^M|| \leq ||v_N^M|| + ||u_N^0||$, we get **Theorem** 4.3. □

Next we prove the convergence of (3.7) and (3.8).

Theorem 4.4. *Let u is the solution of (1.1), and $\{u_N^k\}_{k=0}^M$ are the solutions of (3.7) and (3.8), assume $u \in H^1([0,T]) \times (H^m(\Omega) \times H_0^1(\Omega)), m > 1$, $u^0 = \pi_N^{1,0}u^0$, we have*

$$||u^n - u_N^n|| \leq \tilde{C}\tau^2 + C\tau^{\tilde{\sigma}-\frac{1}{2}} + C\tau^{\sigma+\gamma_1-\frac{1}{2}} + C\tau^{\sigma+\gamma_2-\frac{1}{2}} + CN^{-m}.$$

Proof. For simplicity, define $u^n - u_N^n = (u^n - \pi_N^{1,0}u^n) + (\pi_N^{1,0}u^n - u_N^n) \triangleq \eta^n + e_N^n$, note that $\eta^0 = e_N^0 = 0$, then we have

Case $n = 1$:

$$\begin{aligned}\left(\frac{\eta^1 + e_N^1}{\tau}, v\right) =& -\tau^{\gamma_1-1}\omega_0^{(1-\gamma_1)}\left(\partial_x e_N^1, \partial_x v\right) \\ & -\tau^{\gamma_2-1}\omega_0^{(1-\gamma_2)}\left(\eta^1 + e_N^1, v\right) + \left(E_{1-\theta}, v\right).\end{aligned} \tag{4.13}$$

Case $n \geq 2$:

$$\begin{aligned}&\frac{3-2\theta}{2\tau}\left(\eta^n + e_N^n, v\right) - \frac{2-2\theta}{\tau}(\eta^{n-1} + e_N^{n-1}, v) + \frac{1-2\theta}{2\tau}(\eta^{n-2} + e_N^{n-2}, v) \\ =& -\tau^{\gamma_1-1}\sum_{k=1}^{n}\omega_{n-k}^{(1-\gamma_1)}\left(\partial_x e_N^k, \partial_x v\right) - \tau^{\gamma_2-1}\sum_{k=1}^{n}\omega_{n-k}^{(1-\gamma_2)}\left(\eta^k + e_N^k, v\right) + \left(E_{n-\theta}, v\right).\end{aligned} \tag{4.14}$$

We set v to e_N^n and replace n with j. Summing j from 1 to n ($n \geq 2$), we get

$$\begin{aligned}&\frac{1}{\tau}\left(e_N^1, e_N^1\right) + \frac{3-2\theta}{2\tau}\sum_{j=2}^{n}\left(e_N^j, e_N^j\right) - \frac{2-2\theta}{\tau}\sum_{j=2}^{n}(e_N^{j-1}, e_N^j) \\ &+ \frac{1-2\theta}{2\tau}\sum_{j=2}^{n}(e_N^{j-2}, e_N^j) + \tau^{\gamma_1-1}\sum_{j=1}^{n}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_1)}(\partial_x e_N^k, \partial_x e_N^j) \\ &+ \tau^{\gamma_2-1}\sum_{j=1}^{n}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}(e_N^k, e_N^j) = -\frac{1}{\tau}(\eta^1, e_N^1) - \frac{3-2\theta}{2\tau}\sum_{j=2}^{n}\left(\eta^j, e_N^j\right) \\ &+ \frac{2-2\theta}{\tau}\sum_{j=2}^{n}(\eta^{j-1}, e_N^j) - \frac{1-2\theta}{2\tau}\sum_{j=2}^{n}(\eta^{j-2}, e_N^j) \\ &- \tau^{\gamma_2-1}\sum_{j=1}^{n}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}(\eta^k, e_N^j) + \sum_{j=1}^{n}(E_{j-\theta}, e_N^j).\end{aligned} \tag{4.15}$$

Using Lemmas 4.1 and 4.2, we have the following estimates

$$\begin{aligned}
&\frac{1}{\tau}\left(e_N^1, e_N^1\right) + \frac{3-2\theta}{2\tau}\sum_{j=2}^{n}\left(e_N^j, e_N^j\right) - \frac{2-2\theta}{\tau}\sum_{j=2}^{n}(e_N^{j-1}, e_N^j) \\
&+\frac{1-2\theta}{2\tau}\sum_{j=2}^{n}(e_N^{j-2}, e_N^j) \geq \frac{1}{4\tau}||e_N^n||^2 - \frac{1}{2\tau}||e_N^1||^2, \\
&\tau^{\gamma_1-1}\sum_{j=1}^{n}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_1)}(\partial_x e_N^k, \partial_x e_N^j) \geq 0, \\
&\tau^{\gamma_2-1}\sum_{j=1}^{n}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}(e_N^k, e_N^j) \geq 0.
\end{aligned} \tag{4.16}$$

By (2.1), we can get

$$\eta(t) = (u_0 - \pi_N^{1,0}u_0) + (u_1 - \pi_N^{1,0}u_1)t + \sum_{j=2}^{m}(c_j - \pi_N^{1,0}c_j)t^{\sigma_j} + (\phi - \pi_N^{1,0}\phi), \tag{4.17}$$

and we derive $||\eta_t|| + ||D_{0,t}^{1-\gamma_2}\eta|| \leq CN^{-m}$ based on (2.3). Further, we have

$$D_{\tau,1-\gamma_1}^{n,\theta}\eta - D_{0,t}^{1-\gamma_1}\eta(t_{n-\theta}) = O(t_{n-\theta}^{\sigma+\gamma_1-3}\tau^2). \tag{4.18}$$

$$D_{\tau,1-\gamma_2}^{n,\theta}\eta - D_{0,t}^{1-\gamma_2}\eta(t_{n-\theta}) = O(t_{n-\theta}^{\sigma+\gamma_2-3}\tau^2). \tag{4.19}$$

case $n = 1$:

$$\frac{1}{\tau}\eta^1 - \eta_t(t_{1-\theta}) = O(t_{1-\theta}^{\tilde{\sigma}-3}\tau^2), \tag{4.20}$$

case $n \geq 2$:

$$\frac{3-2\theta}{2\tau}\eta^n - \frac{2-2\theta}{\tau}\eta^{n-1} + \frac{1-2\theta}{2\tau}\eta^{n-2} - \eta_t(t_{n-\theta}) = O(t_{n-\theta}^{\tilde{\sigma}-3}\tau^2), \tag{4.21}$$

We konw that

$$\tau\sum_{j=1}^{n}t_{j-\theta}^{s} = \begin{cases} O(1), & s > -1 \\ O(\log n), & s = -1 \\ O(\tau^{1+s}), & s < -1 \end{cases} \tag{4.22}$$

combining (4.18)-(4.22), we derive

$$\begin{aligned}
&\tau\sum_{j=1}^{n}||D_{\tau,1-\gamma_1}^{j,\theta}\eta - D_{0,t}^{1-\gamma_1}\eta(t_{j-\theta})||^2 \qquad (4.23)\\
&\leq \tilde{E}_{n-\theta}^{(1)} \triangleq C\tau^5\sum_{j=1}^{n}t_{j-\theta}^{2\sigma+2\gamma_1-6} = \begin{cases} O(\tau^4), & \sigma > -\gamma_1 + 2.5 \\ O(\tau^4\log n), & \sigma = -\gamma_1 + 2.5 \\ O(\tau^{2\sigma+2\gamma_1-1}), & \sigma < -\gamma_1 + 2.5 \end{cases}
\end{aligned}$$

$$\tau\sum_{j=1}^{n}||D_{\tau,1-\gamma_2}^{j,\theta}\eta-D_{0,t}^{1-\gamma_2}\eta(t_{j-\theta})||^2 \tag{4.24}$$
$$\leq\tilde{E}_{n-\theta}^{(2)}\triangleq C\tau^5\sum_{j=1}^{n}t_{j-\theta}^{2\sigma+2\gamma_2-6}=\begin{cases}O(\tau^4), & \sigma>-\gamma_2+2.5\\ O(\tau^4\log n), & \sigma=-\gamma_2+2.5\\ O(\tau^{2\sigma+2\gamma_2-1}), & \sigma<-\gamma_2+2.5\end{cases}$$

$$\tau||\frac{1}{\tau}\eta^1-\eta_t(t_{1-\theta})||^2+\tau\sum_{j=2}^{n}||\frac{3-2\theta}{2\tau}\eta^j-\frac{2-2\theta}{\tau}\eta^{j-1}+\frac{1-2\theta}{2\tau}\eta^{j-2}-\eta_t(t_{j-\theta})||^2$$
$$\leq\tilde{E}_{n-\theta}^{(3)}\triangleq C\tau^5\sum_{j=1}^{n}t_{j-\theta}^{2\tilde{\sigma}-6}=\begin{cases}O(\tau^4), & \tilde{\sigma}>2.5\\ O(\tau^4\log n), & \tilde{\sigma}=2.5\\ O(\tau^{2\tilde{\sigma}-1}), & \tilde{\sigma}<2.5\end{cases} \tag{4.25}$$

Multiplying by τ both sides of (4.15), we have

$$\tau\left(\frac{1}{\tau}(\eta^1,e_N^1)+\frac{3-2\theta}{2\tau}\sum_{j=2}^{n}\left(\eta^j,e_N^j\right)-\frac{2-2\theta}{\tau}\sum_{j=2}^{n}(\eta^{j-1},e_N^j)+\frac{1-2\theta}{2\tau}\sum_{j=2}^{n}(\eta^{j-2},e_N^j)\right)$$
$$\leq\frac{\tau}{2}\left(||\frac{1}{\tau}\eta^1||^2+\sum_{j=2}^{n}||\frac{3-2\theta}{2\tau}\eta^j-\frac{2-2\theta}{\tau}\eta^{j-1}+\frac{1-2\theta}{2\tau}\eta^{j-2}||^2\right)+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2$$
$$\leq\tau\left(||\frac{1}{\tau}\eta^1-\eta_t(t_{1-\theta})||^2+||\eta_t(t_{1-\theta})||^2\right)$$
$$+\tau\sum_{j=2}^{n}\left(||\frac{3-2\theta}{2\tau}\eta^j-\frac{2-2\theta}{\tau}\eta^{j-1}+\frac{1-2\theta}{2\tau}\eta^{j-2}-\eta_t(t_{n-\theta})||^2+||\eta_t(t_{n-\theta})||^2\right) \tag{4.26}$$
$$+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2\leq\tilde{E}_{n-\theta}^{(3)}+CN^{-2m}+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2,$$

$$\tau\sum_{j=1}^{n}\left(\tau^{\gamma_2-1}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}(\eta^k,e_N^j)\right)$$
$$\le\frac{\tau}{2}\sum_{j=1}^{n}||\tau^{\gamma_2-1}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}\eta^k||^2+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2$$
$$\le\tau\sum_{j=1}^{n}\left(||\tau^{\gamma_2-1}\sum_{k=1}^{j}\omega_{j-k}^{(1-\gamma_2)}\eta^k-D_{0,t}^{1-\gamma_2}\eta(t_{j-\theta})||^2+||D_{0,t}^{1-\gamma_2}\eta(t_{j-\theta})||^2\right)+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2$$
$$\le\tilde{E}_{n-\theta}^{(2)}+CN^{-2m}+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2, \tag{4.27}$$

$$\tau\sum_{j=1}^{n}(E_{j-\theta},e_N^j)\le\tilde{E}_{n-\theta}^{(1)}+\tilde{E}_{n-\theta}^{(2)}+\tilde{E}_{n-\theta}^{(3)}+\frac{\tau}{2}\sum_{j=1}^{n}||e_N^j||^2. \tag{4.28}$$

Combining (4.16), (4.26)-(4.28), when $n\ge 2$ we get

$$\frac{1}{4}||e_N^n||^2\le\frac{1}{2}||e_N^1||^2+\tilde{E}_{n-\theta}^{(1)}+\tilde{E}_{n-\theta}^{(2)}+\tilde{E}_{n-\theta}^{(3)}+CN^{-2m}+\frac{3}{2}\tau\sum_{j=1}^{n}||e_N^j||^2. \tag{4.29}$$

Similarly, when $n=1$ we can easily get the following estimates

$$||e_N^1||^2\le\tilde{E}_{1-\theta}^{(1)}+\tilde{E}_{1-\theta}^{(2)}+\tilde{E}_{1-\theta}^{(3)}+CN^{-2m}. \tag{4.30}$$

Using Grönwall's inequality (Lemma 2.2), we obtain

$$||e_N^n||^2\le\tilde{C}\tau^4+C\tau^{2\tilde{\sigma}-1}+C\tau^{2\sigma+2\gamma_1-1}+C\tau^{2\sigma+2\gamma_2-1}+CN^{-2m}, \tag{4.31}$$

where C is independent of n and τ. $\tilde{C}$ is defined by

$$\tilde{C}=\begin{cases}O(\sqrt{\log n}), & \tilde{\sigma}=2.5,\ \sigma=-\gamma_1+2.5,\ \text{and}\ \sigma=-\gamma_2+2.5\\ O(1), & \text{otherwise.}\end{cases} \tag{4.32}$$

Finally, by using triangle inequality and (2.3), we get **Theorem** 4.4. □

5. Analysis for non-smooth solutions

The solutions of FDEs are not generally regular with respect to time. We can see from (3.6), in the case of $\sigma<3$, the discretization of (3.4) and (3.5) can not obtain second order convergence in time. To obtain the optimal convergence rate in this case, we refer [12, 35] and there are some other interesting approaches to deal with the problems with non-smooth solutions, interested readers are referred to references [38, 39]. We rewrite the approximation of (3.1) and (3.3) as follows

$$\begin{cases} D_{0,t}^{\gamma}u(t_{n-\theta}) \approx D_{\tau,\gamma}^{n,\theta}u + \tau^{-\gamma}\sum_{j=1}^{m} w_{n,j}^{(\gamma)}(u^j - u^0), \\ u_t(t_{n-\theta}) \approx u_{\tau,\theta}^{n} + \tau^{-1}\sum_{j=1}^{m} w_{n,j}^{(1)}(u^j - u^0), \end{cases} \tag{5.1}$$

where $w_{n,j}^{(\gamma)}$, $w_{n,j}^{(1)}$ are starting weights. Above approximation holds exactly for $u(t) = t^{\sigma_r}$ with $\sigma_r < 3$ and the starting weights can be derived by solving following linear systems respectively.

$$\begin{cases} \sum_{j=1}^{m} w_{n,j}^{(\gamma)} t_j^{\sigma_r} = \tau^{\gamma}\dfrac{\Gamma(\sigma_r+1)}{\Gamma(\sigma_r+1-\gamma)} t_{n-\theta}^{\sigma_r-\gamma} - \sum_{k=1}^{n} \omega_{n-k}^{(\gamma)} t_k^{\sigma_r}, \\ \sum_{j=1}^{m} w_{n,j}^{(1)} t_j^{\sigma_r} = \tau\sigma_r t_{n-\theta}^{\sigma_r-1} - t_1^{\sigma_r}, & n=1 \\ \sum_{j=1}^{m} w_{n,j}^{(1)} t_j^{\sigma_r} = \tau\sigma_r t_{n-\theta}^{\sigma_r-1} - \dfrac{3-2\theta}{2} t_n^{\sigma_r} + (2-2\theta) t_{n-1}^{\sigma_r} - \dfrac{1-2\theta}{2} t_{n-2}^{\sigma_r}, & n \geq 2 \end{cases} \tag{5.2}$$

We rewrite fully discrete scheme of (1.1) with the approximation (5.1) with correction terms

Case $n = 1$:

$$\begin{aligned} &\left(\frac{u_N^1 - u_N^0}{\tau}, v\right) + \frac{1}{\tau}\sum_{j=1}^{m} w_{n,j}^{(1)}(u_N^j - u_N^0, v) \\ =& -\tau^{\gamma_1-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_1)}\left(\partial_x u_N^k - \partial_x u_N^0, \partial_x v\right) - \tau^{\gamma_1-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_1)}(\partial_x u_N^j - \partial_x u_N^0, \partial_x v) \\ & -\tau^{\gamma_2-1}\sum_{k=0}^{1}\omega_{1-k}^{(1-\gamma_2)}\left(u_N^k - u_N^0, v\right) - \tau^{\gamma_2-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_2)}(u_N^j - u_N^0, v) + \left(f^{1-\theta}, v\right), \end{aligned} \tag{5.3}$$

Case $n \geq 2$:

$$\begin{aligned} &\frac{3-2\theta}{2\tau}\left(u_N^n, v\right) - \frac{2-2\theta}{\tau}\left(u_N^{n-1}, v\right) + \frac{1-2\theta}{2\tau}\left(u_N^{n-2}, v\right) + \frac{1}{\tau}\sum_{j=1}^{m} w_{n,j}^{(1)}(u_N^j - u_N^0, v) \\ =& -\tau^{\gamma_1-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_1)}\left(\partial_x u_N^k - \partial_x u_N^0, \partial_x v\right) - \tau^{\gamma_1-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_1)}(\partial_x u_N^j - \partial_x u_N^0, \partial_x v) \\ & -\tau^{\gamma_2-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma_2)}\left(u_N^k - u_N^0, v\right) - \tau^{\gamma_2-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_2)}(u_N^j - u_N^0, v) + \left(f^{n-\theta}, v\right). \end{aligned} \tag{5.4}$$

Remark 2. Adding correction terms don't affect the stability. We just need to move the adding terms to the right hand of the equation (5.3) and (5.4), replace $f^{n-\theta}$ with $f^{n-\theta}-\frac{1}{\tau}\sum_{j=1}^{m} w_{n,j}^{(1)}(u_N^j-u_N^0)+\tau^{\gamma_1-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_1)}\Delta(u_N^j-u_N^0)-\tau^{\gamma_2-1}\sum_{j=1}^{m} w_{n,j}^{(1-\gamma_2)}(u_N^j-u_N^0)$, then the proof of stability is the same as the proof in Theorem 4.3.

6. Fast algorithm

The coefficients $\omega_n^{(\gamma)}$ can be expressed as integrals by

$$\omega_n^{(\gamma)} = \frac{\tau^{\gamma+1}}{2\pi i}\int_{\Gamma} e_n(\tau\lambda)F_\gamma(\lambda)d\lambda, \tag{6.1}$$

where $e_n(z) = q_0(z)r_0(z)^n,\ F_\gamma(\lambda) = \lambda^\gamma,\ q_0(z) = \frac{1}{1-z}$ and $r_0(z) = \frac{1}{1-z}$ [12]. Given a base B, B is a positive integer and $B > 1$, we split the time domain into a sequence of fast growing intervals :

$$I_l = [B^{l-1}\tau, (2B^l - 2)\tau]. \tag{6.2}$$

To approximate the weights, we choose a Talbot contour Γ [36]

$$(-\pi,\pi) \to \Gamma : \vartheta \mapsto \varrho(\vartheta) = \frac{K}{T_l}((\vartheta cot(\vartheta) + ik\vartheta)v + \sigma), \tag{6.3}$$

see the Fig. 1(a). By (6.1), we can obtain

$$\omega_n^{(\gamma)} \approx \tau^{\gamma+1}\sum_{j=-K}^{K} w_j^{(l)} e_n(\tau\lambda_j^{(l)})F_\gamma(\lambda_j^{(l)}),\ n\tau \in I_l \tag{6.4}$$

where the weights $w_j^{(l)}$ and quadrature points $\lambda_j^{(l)}$ are given by

$$w_j^{(l)} = -\frac{i}{2(K+1)}\varrho'(\vartheta_j),\ \lambda_j^{(l)} = \varrho(\vartheta_j),\ \vartheta_j = \frac{j\pi}{K+1}. \tag{6.5}$$

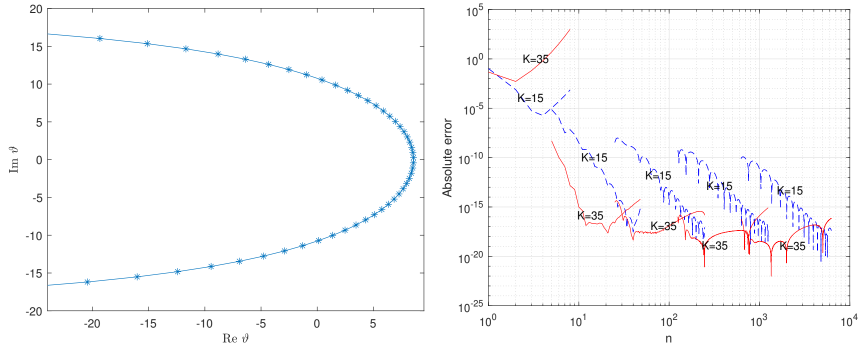
By subtracting (6.4) from (3.2) and taking the absolute value, we obtain their absolute approximation error. We take $B = 5, I_l(l = 1, 2, 3, 4, 5)$, and $K = 15$ and 35, respectively. The absolute approximation error is shown in Fig. 1(b).

Fig. 1(b) shows the approximation using (6.4) for first few weights has large error, so we need to consider the formula (3.2) for calculating the first few weights.

Firstly we choose suitable points b_l $(l = 0, 1, \cdots, L)$ to split (3.1) into $L + 1$ sums, where L is the smallest integer satisfying $n + 2 \le 2B^L$, and $n = b_0 > b_1 > \cdots > b_{L-1} > b_L = 0$.

$$D_{\tau,\gamma}^{n,\theta} = \tau^{-\gamma}\omega_0^{(\gamma)}(u^n - u^0) + \tau^{-\gamma}\sum_{l=1}^{L}\sum_{k=b_l}^{b_{l-1}-1}\omega_{n-k}^{(\gamma)}(u^k - u^0). \tag{6.6}$$

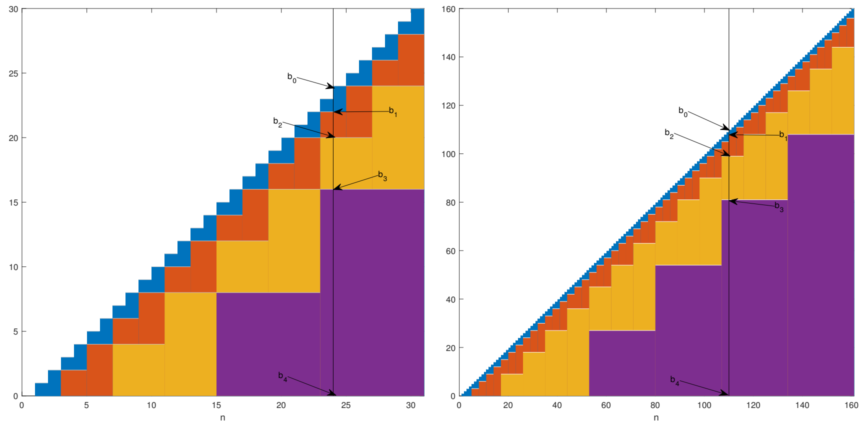
We have $[(n - b_{l-1} + 1)\tau, (n - b_l)\tau] \subset I_l$ and b_l $(l = 0, 1, \cdots, L)$ add B^l every B^l steps. Given $B = 2$ and $B = 3$, the relation between b_l and I_l



(a) (b)

Fig. 1. (a) Talbot contour, (b) Absolute approximation error.

is indicated in Fig. 2, b_l is the ordinate where the solid line intersects with different colored regions.



(a) (b)

Fig. 2. (a) For $B = 2$, $(b_0, b_1, b_2, b_3, b_4)$=$(24, 22, 20, 16, 0)$, (b) For $B = 3$, $(b_0, b_1, b_2, b_3, b_4)$=$(110, 108, 99, 81, 0)$.

For more detailed information on points b_l, we refer readers to the reference [37].

For convenience, we define $v_{n,\gamma}^{(l)}$ as

$$v_{n,\gamma}^{(l)} = \begin{cases} \tau^{-\gamma}\omega_0^{(\gamma)}(u^n - u^0), & l = 0, \\ \tau^{-\gamma}\displaystyle\sum_{k=b_l}^{b_{l-1}-1} \omega_{n-k}^{(\gamma)}(u^k - u^0), & l = 1, 2, \cdots, L. \end{cases} \tag{6.7}$$

Next, combining (6.4), (6.7) and the definition for $e_n(z)$, we have (for $l > 0$)

$$\begin{aligned} v_{n,\gamma}^l &\approx \sum_{j=-K}^{K} w_j^{(l)} \left[\tau \sum_{k=b_l}^{b_{l-1}-1} e_{n-k}(\tau\lambda_j^{(l)})(u^k - u^0) \right] F_\gamma(\lambda_j^{(l)}) \\ &= \sum_{j=-K}^{K} w_j^{(l)} r_0^{n-(b_{l-1}-1)}(\tau\lambda_j^{(l)}) y_j F_\gamma(\lambda_j^{(l)}), \end{aligned} \tag{6.8}$$

where y_j is given following

$$y_j = y_j(b_l, b_{l-1}, \lambda_j^{(l)}) = \tau \sum_{k=b_l}^{b_{l-1}-1} e_{(b_{l-1}-1)-k}(\tau\lambda_j^{(l)})(u^k - u^0). \tag{6.9}$$

Because $y_j(b_l, b_{l-1}, \lambda_j^{(l)})$ has a recursive structure, we can make use of it to accelerate computation further,

$$\begin{aligned} y_j(b_l, b_s, \lambda_j^{(l)}) =& \tau \sum_{k=b_l}^{b_m-1} e_{(b_s-1)-k}(\tau\lambda_j^{(l)})(u^k - u^0) + y_j(b_m, b_s, \lambda_j^{(l)}) \\ =& r_0(\tau\lambda_j^{(l)})^{b_s-b_m} y_j(b_l, b_m, \lambda_j^{(l)}) + y_j(b_m, b_s, \lambda_j^{(l)}). \end{aligned} \tag{6.10}$$

We first store y_j of the last time step for $l = 1, 2, \cdots, L$, then we judge whether y_j has been stored when we calculate next time step. Give an example to describe it, for $B = 3$ and $n = 110$, as depicted in Fig. 2(b), we store

$$y_j(108, 110, \lambda_j^{(1)}),\ y_j(99, 108, \lambda_j^{(2)}),\ y_j(81, 99, \lambda_j^{(3)}),\ y_j(0, 81, \lambda_j^{(4)}), \tag{6.11}$$

then, for $n = 111$, we need to calculate

$$y_j(108, 111, \lambda_j^{(1)}),\ y_j(99, 108, \lambda_j^{(2)}),\ y_j(81, 99, \lambda_j^{(3)}),\ y_j(0, 81, \lambda_j^{(4)}), \tag{6.12}$$

and we can find only the first term has not stored. So we need merely to calculate the first term for $n = 111$ to accelerate the computation.

The first few weights obtained by (6.4) are poor (see Fig. 2(b)), For $l = 0, 1, 2, \cdots, k$, we use (3.2) to calculate the weights and for $l = k+1, \cdots, L$, we use (6.4) to calculate the weights. Combine (6.6),(6.7) and (6.8),we derive

$$\begin{aligned} D_{\tau,\gamma}^{n,\theta} u &= \sum_{l=0}^{k} v_{n,\gamma}^{(l)} + \sum_{l=k+1}^{L} v_{n,\gamma}^{(l)} \\ \approx& \sum_{l=0}^{k} v_{n,\gamma}^{(l)} + \sum_{l=k+1}^{L} \sum_{j=-K}^{K} w_j^{(l)} r_0^{n-(b_{l-1}-1)}(\tau\lambda_j^{(l)}) y_j F_\gamma(\lambda_j^{(l)}). \end{aligned} \tag{6.13}$$

Using (6.13), we can easily rewrite the fully discrete scheme with the fast algorithm.

7. Numerical experiments

In this section, we design four cases to verify effectiveness of our method in solving the fractional cable equation. The basis function is $\phi(x) = L_j(x) - L_{j+2}(x),\ j = 0, 1, \cdots, N$. For $\forall v_N^k \in P_N^0,\ v_N^k = \sum_{j=0}^{N-2} \hat{v}_N^k \phi_j(x)$, where $\hat{v}_N^k$ is the frequency coefficient.

Case 1: We consider (1.1) with homogeneous initial condition $\Phi_0(x) = 0,\ \Phi_1(x) = 0$. Its exact solution is $u(x,t) = t^4 \sin(\pi x)$ and the forcing term is $f(x,t) = (4t^3 + \frac{\pi^2\Gamma(5)}{\Gamma(4+\gamma_1)} t^{3+\gamma_1} + \frac{\Gamma(5)}{\Gamma(4+\gamma_2)} t^{3+\gamma_2}) \sin(\pi x)$. Let $N = 100,\ T = 1,\ L = 1$ and the results are indicated in Table 1-3.

Table 1. Errors and temporal convergence orders with $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, and $\theta = 0.2$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
2^{-6}	1. 091633e-03		1. 846247e-04		0. 245754
2^{-7}	2. 744767e-04	1. 99	4. 642145e-05	1. 99	0. 672443
2^{-8}	6. 881590e-05	2. 00	1. 163864e-05	2. 00	2. 577600
2^{-9}	1. 722861e-05	2. 00	2. 913825e-06	2. 00	11. 333679
2^{-10}	4. 310235e-06	2. 00	7. 289776e-07	2. 00	55. 777426

Table 2. Errors and temporal convergence orders with $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, and $\theta = 0.5$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
2^{-6}	2. 352760e-03		3. 979155e-04		0. 236521
2^{-7}	5. 918420e-04	1. 99	1. 000965e-04	1. 99	0. 737517
2^{-8}	1. 484186e-04	2. 00	2. 510160e-05	2. 00	2. 701757
2^{-9}	3. 716199e-05	2. 00	6. 285101e-06	2. 00	11. 047976
2^{-10}	9. 297673e-06	2. 00	1. 572489e-06	2. 00	55. 935405

Table 3. Errors and temporal convergence orders with $\gamma_1 = 0.8$, $\gamma_2 = 0.7$, and $\theta = 0.9$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
2^{-6}	1. 077429e-02		1. 822225e-03		0. 247115
2^{-7}	2. 730838e-03	1. 99	4. 618588e-04	1. 98	0. 708011
2^{-8}	6. 873984e-04	2. 00	1. 162577e-04	2. 00	2. 612621
2^{-9}	1. 724376e-04	2. 00	2. 916388e-05	2. 00	11. 671898
2^{-10}	4. 318303e-05	2. 00	7. 303423e-06	2. 00	53. 581065

To inspect the spatial accuracy, we take $\tau = 0.001$ to eliminate the error in temporal direction. Fig. 3 shows that when $\gamma_1 = 0.3, \gamma_2 = 0.7, \theta = 0.5$, the error decreases exponentially, that is, the spectral accuracy, which verifies our

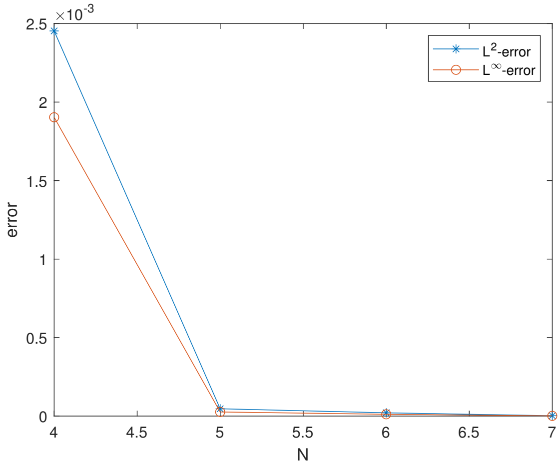


Fig. 3. $\gamma_1 = 0.3, \gamma_2 = 0.7, \theta = 0.5$ for Example 1 at $T = 1$

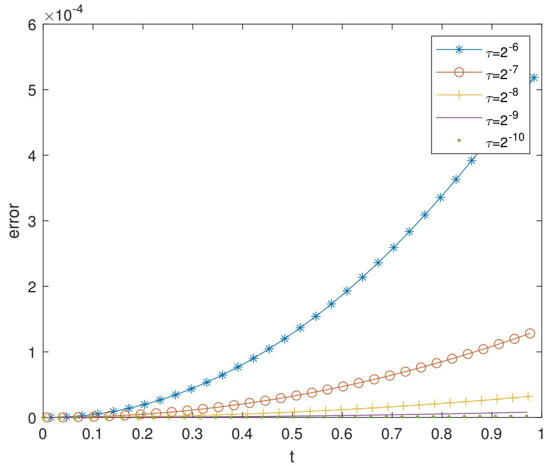


Fig. 4. L^∞ error for Example 1 at $N = 7$.

theoretical analysis. Fig. 4 shows the L^∞ error at $T = 1, N = 7$ for Example 1.

Case 2: We consider the (1.1) with non-homogeneous initial condition $\Phi_0(x) = \sin(\pi x),\ \Phi_1(x) = 0$. Its exact solution is $u(x,t) = (t^4+1)\sin(\pi x)$ and the forcing term is $f(x,t) = (4t^3 + \frac{\pi^2\Gamma(5)}{\Gamma(4+\gamma_1)}t^{3+\gamma_1} + \frac{\Gamma(5)}{\Gamma(4+\gamma_2)}t^{3+\gamma_2})\sin(\pi x)$. Let $N = 100,\ T = 1$ and $L = 1$. The responding results are shown in Table 4-6. We can see that the numerical scheme is still applicable to the equations with non-homogeneous initial conditions, which reflects the stability of our method.

Table 4. Errors and temporal convergence orders with $\gamma_1 = 0.7$, $\gamma_2 = 0.8$, and $\theta = 0.6$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
1/40	4. 038357e-03		2. 152599e-03		0. 050131
1/80	1. 025594e-03	1. 98	5. 466764e-04	1. 98	0. 134863
1/160	2. 584151e-04	1. 99	1. 377415e-04	1. 99	0. 447734
1/320	6. 485714e-05	1. 99	3. 456890e-05	1. 99	2. 270288
1/640	1. 624621e-05	2. 00	8. 658372e-06	2. 00	14. 674638

Table 5. Errors and temporal convergence orders with $\gamma_1 = 0.7$, $\gamma_2 = 0.6$, and $\theta = 0.5$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
1/40	2. 866934e-03		1. 528181e-03		0. 045444
1/80	7. 266525e-04	1. 98	3. 873282e-04	1. 98	0. 135609
1/160	1. 829122e-04	1. 99	9. 749519e-05	1. 99	0. 407822
1/320	4. 588503e-05	2. 00	2. 445592e-05	2. 00	2. 216844
1/640	1. 149112e-05	2. 00	6. 123674e-06	2. 00	15. 022102

Table 6. Errors and temporal convergence orders with $\gamma_1 = 0.4$, $\gamma_2 = 0.4$, and $\theta = 0.3$.

τ	$\|\|E(N,\tau)\|\|_2$	Order	$\|\|E(N,\tau)\|\|_\infty$	Order	Time(s)
1/40	1. 821326e-04		9. 704045e-05		0. 019041
1/80	4. 509651e-05	2. 02	2. 400473e-05	2. 02	0. 116742
1/160	1. 122625e-05	2. 01	5. 957047e-06	2. 01	0. 467209
1/320	2. 812157e-06	2. 01	1. 474675e-06	2. 01	2. 243742
1/640	7. 264002e-07	2. 03	3. 600262e-07	2. 03	13. 807315

Case 3: We consider the solution of (1.1) is non-smooth. Let (1.1) with homogeneous initial condition $\Phi_0(x) = 0,\ \Phi_1(x) = 0$. Its exact solution is $u(x,t) = (t^4 + t^{\frac{3}{2}})\sin(\pi x)$, which is non-smooth, and the responding forcing term is $f(x,t) = (4t^3 + \frac{\pi^2\Gamma(5)}{\Gamma(4+\gamma_1)}t^{3+\gamma_1} + \frac{\Gamma(5)}{\Gamma(4+\gamma_2)}t^{3+\gamma_2} + \frac{3}{2}t^{\frac{1}{2}} + \frac{\pi^2\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})+\gamma_1}t^{\frac{1}{2}+\gamma_1} + \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2}+\gamma_2)}t^{\frac{1}{2}+\gamma_2})\sin(\pi x)$. Let $N = 100,\ T = 1$, and $L = 1$. Table 7-9 compare the results using (3.7), (3.8) and (5.3), (5.4). Because the solution is weak regular, the convergence rate is not optimal and with correction terms it is better than the results of scheme (3.7), (3.8). By adding the correction terms, we can overcome the weak regularity and get the optimal convergence rate, which reflects the efficiency of our method.

Table 7. Temporal convergence orders with $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, and $\theta = 0.3$.

τ	Standard	Order	Correction	Order
1/20	6. 751578e-04		5. 708209e-04	
1/40	1. 915618e-04	1. 82	1. 497156e-04	1. 93
1/80	5. 464577e-05	1. 81	3. 835545e-05	1. 96
1/160	1. 587941e-05	1. 78	9. 661586e-06	1. 99
1/320	4. 731055e-06	1. 75	2. 391077e-06	2. 01

Table 8. Temporal convergence orders with $\gamma_1 = 0.3$, $\gamma_2 = 0.4$, and $\theta = 0.2$.

τ	Standard	Order	Correction	Order
1/20	1. 447964e-03		1. 306379e-03	
1/40	3. 829940e-04	1. 92	3. 295222e-04	1. 99
1/80	1. 017298e-04	1. 91	8. 223244e-05	2. 00
1/160	2. 743734e-05	1. 89	2. 049203e-05	2. 00
1/320	7. 578011e-06	1. 86	5. 144713e-06	1. 99

Table 9. Temporal convergence orders with $\gamma_1 = 0.6$, $\gamma_2 = 0.8$, and $\theta = 0.2$.

τ	Standard	Order	Correction	Order
1/20	1. 251645e-04		6. 758207e-05	
1/40	4. 632530e-05	1. 43	2. 660304e-05	1. 35
1/80	1. 646001e-05	1. 49	8. 013154e-06	1. 73
1/160	5. 815174e-06	1. 50	2. 084598e-06	1. 94
1/320	2. 057686e-06	1. 50	5. 163438e-07	2. 01

Case 4: We consider using fast algorithm to solve the (1.1). Let (1.1) with homogeneous initial condition $\Phi_0(x) = 0,\ \Phi_1(x) = 0$. Its exact solution and the responding forcing term are the same as with the first case.

We apply fast algorithm for solving (1.1) and choose $B = 5$ as our basis. Let $N = 100,\ T = 1$ and $L = 1$. For convenience, we use the symbol Fast_K for (6.4) with $2K+1$ approximation points. U_S denote the solutions obtained by the standard method and U_F denote the solutions obtained by the fast algorithm. We take $\gamma = \gamma_1 = \gamma_2 = \frac{1-\theta}{2}$ and the pointwise error is

$$E(\gamma, M) = \max_{t=t_0,\cdots,t_M, x=x_1,\cdots,x_N} |U_S - U_F|. \tag{7.1}$$

As shown in Table 10, when $K = 15$, the pointwise error is about 10^{-7}, and in the case of the same numbers of time stepping, the fast algorithm can save significantly computation time. When $K = 35$, the pointwise error is about 10^{-14} which indicates our method can get high accuracy while saving computation time. We plot the exact solution and the numerical solution in $M = 1000,\ \gamma = 0.5,\ \theta = 0.25$ and $K = 35$, as shown in Fig. 5(a) and Fig.

Table 10. Pointwise error with $\gamma = \gamma_1 = \gamma_2$, and $\theta = \frac{1-\gamma}{2}$.

γ	M	Standard	Fast_{15}	$E(\gamma, M)$	Fast_{35}	$E(\gamma, M)$
0.5	1×10^3	20. 78s	8. 37s	1. 56782E-07	17. 28s	6. 09513E-14
	2×10^3	161. 94s	25. 01s	2. 51764E-07	48. 44s	1. 34559E-13
	3×10^3	568. 29s	42. 01s	3. 24574E-07	83. 51s	1. 86859E-13
	4×10^3	1305. 80s	61. 43s	3. 97923E-07	116. 53s	1. 92960E-13
0.8	1×10^3	20.72s	8.47s	8.88695E-08	17.28s	2.00950E-14
	2×10^3	162.69s	21.85s	8.68453E-08	44.98s	5.73988E-14
	3×10^3	560.18s	39.84s	9.79511E-08	77.72s	4.51867E-14
	4×10^3	1414.12s	59.53s	8.64079E-08	116.06s	6.96125E-14
0.2	1×10^3	21.88s	9.27s	1.09855E-06	17.11s	8.07179E-14
	2×10^3	160.17s	22.36s	1.93400E-06	52.25s	1.84527E-13
	3×10^3	559.55s	43.46s	2.66548E-06	88.96s	3.58885E-13
	4×10^3	1303.94s	67.39s	3.30004E-06	128.67s	3.67707E-13

5(b), the corresponding error contour is shown in Fig. 5(c). To see the time complexity of the fast algorithm, we plot the double logarithmic chart, see Fig. 5(d). We can clearly see the fast algorithm have reduced computation work from $O(M^2)$ to $O(M \log M)$.

8. Conclusion

In this study, we have developed a fast time stepping Legendre spectral method for solving the fractional Cable equation. We analyze the stability and convergence of our method. For non-smooth solutions, correction terms are considered. In addition, we accelerate the calculation by the fast algorithm. Numerical experiments confirm efficiency and less time-consumption of our method. Our method can be extended to solve other fractional subdiffusion equations and fractional wave equations, and it is feasible to solve higher dimensions cases.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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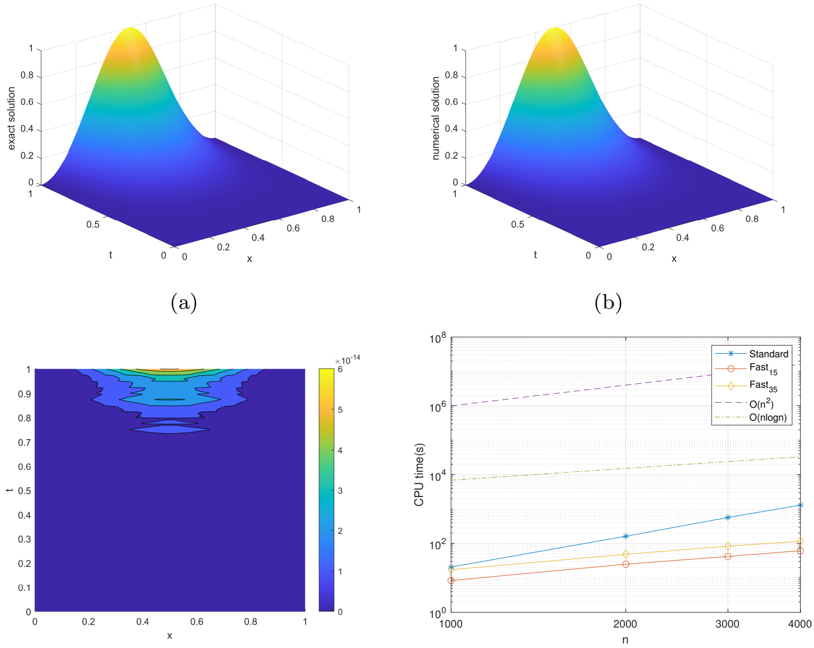

Fig. 5. (a) Exact solution, (b) Numerical solution, (c) Contours of pointwise error, (d) CPU time for standard and fast algorithm.

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